Between Infinitary Logic and Abstract Elementary Classes

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Barranquilla - Universidad del Norte - mayo/junio de 2018
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LONG STORY SHORT

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After many attempts, the analysis of that primal question ran off from the syntactic extreme (infinitary logic(s)) to a more semantic “extreme”. The attempts:

▶ (Keisler) Use “sequentially homogeneous” models. But sequential homogeneity is a consequence of categoricity...

▶ (Shelah) The role of models of size $\aleph_n (n < \omega)$ in the decomposition of large models, the role of dimension-like obstructions.

▶ (Shelah) Forcing-like approach to types that would eventually become “Galois types”.
Algebraically-minded model theory

Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah’s idea of (as expressed in his paper *The Lazy Model-Theoretician’s Guide to Stability Theory* 1973)
Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah’s idea of (as expressed in his paper *The Lazy Model-Theoretician’s Guide to Stability Theory* 1973) speaking mainly to “those who are interested in algebraically-minded model theory, i.e., generic models, the class of e-closed models and universal-homogeneous models rather than elementary classes and saturated models. These were his words in 1975. He continues: “our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions).
WHAT GOES THROUGH, REALLY?

This declaration (the “almost everything goes through”) entailed more than it could seem at first sight: in many ways it is true but it took a long time to build up the right notions of stability, of types, of independence.
Replacing formulas by an abstract notion of “strong embedding” between $L$-structures is the first important point. In the definition of AECs we do not declare membership in the class by satisfying some sentence or some axiomatic system. The relation $|=,$ basic in First Order logic, takes a back seat here, and the main relation $\leq_K$ (a generalization of the elementary submodel relation $\prec$ of first order) now leads the game.
Kennedy’s Formalism Freeness

All of this approach very much goes in line with other situations in mathematics where versions of “Formalism Freeness” (Kennedy) take up center stage. One of them is computability (Turing, Post, Gödel, Kleene, Church). Another one is Model Theory as a “generalized Galois theory”, as happens in AECs.
The definition [Abstract Elementary Class]

Fix a language $L$. A class $\mathcal{K}$ of $L$-structures, together with a binary relation $\leq_K$ on $\mathcal{K}$ is an abstract elementary class (for short, AEC) if:

1. Both $\mathcal{K}$ and $\leq_K$ are closed under isomorphism. This means two things: first, if $M' \cong M \in \mathcal{K}$ then $M' \in \mathcal{K}$; second, if $M', N'$ are $L$-structures with $M' \subseteq N'$, $M' \cong M$, $N' \cong N$ and $M \leq_K N$ then $M' \leq_K N'$.

2. If $M, N \in \mathcal{K}$, $M \leq_K N$ then $M \subseteq N$,

3. $\leq_K$ is a partial order,

4. (Coherence) If $M \subseteq N \leq_K N'$ and $M \leq_K N'$ then $M \leq_K N$,

5. (LS) There is a cardinal (called “the Löwenheim-Skolem number” of the class) $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that if $M \in \mathcal{K}$ and $A \subseteq |M|$, then there is $N \leq_K M$ with $A \subseteq |N|$ and $|N| \leq |A| + LS(\mathcal{K})$.

6. (Unions of $\leq_K$-chains) If $(M_i)_{i<\delta}$ is a $\leq_K$-increasing chain of length $\delta$ ($\delta$ a limit ordinal), then

- $\bigcup_{i<\delta} (M_i)_{i<\delta} \in \mathcal{K}$,
- for each $j < \delta$, $M_j \leq_K \bigcup_{i<\delta} M_i$,
- if for each $i < \delta$, $M_i \leq_K N \in \mathcal{K}$ then $\bigcup_{i<\delta} M_i \leq_K N$. 


MAIN CONJECTURE: THE MAIN GAP

The Main Gap Theorem for FO logic

(Saharon Shelah, c. 1980)
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The Main Gap Theorem for FO logic

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The “gold standard” of mathematical logic, of model theory, in various ways, and the main conjecture in AECs.
At this point, we have the following situation:

- So far, no control on possible axiomatization of the class $\mathcal{K}$. The emphasis is placed on its being closed under the constructions specified in the axioms. However, later (in subsection) we focus on the logical control of these classes. Remember Shelah’s “algebraically-minded model theory”.

- These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.
A “taxonomy” of classes of structures.
# Dividing Lines

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Theorem (Shelah)

Let \((\mathcal{K}, \leq_K)\) be an AEC in a language \(L\). Then there exist

- A language \(L' \supset L\), with size \(LS(\mathcal{K})\),
- A (first order) theory \(T'\) in \(L'\) and
- A set of \(T'\)-types, \(\Gamma'\), such that

\[
\mathcal{K} = PC(L, T', \Gamma') := \{M' \upharpoonright L \mid M' \models T', M' \text{ omits } \Gamma'\}.
\]

Moreover, if \(M', N' \models T'\), they both omit \(\Gamma'\), \(M = M' \upharpoonright L\) and \(N = N' \upharpoonright L\),

\[
M' \subset N' \iff M \leq_K N.
\]
Corollary ("Hanf" number of an AEC)

If an AEC $\mathcal{K}$ has a model of cardinality $\geq \beth_{(2^{LS}(\mathcal{K})^+}$ then it has arbitrarily large models.
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Proof: Use the Hanf number for PC classes (this uses the undefinability of well orders - Lessmann, building on Väänänen and Shelah’s earlier arguments).

\[ \square \]

Theorem (Shelah)

Let $(\mathcal{K}, \leq_\mathcal{K})$ be an AEC with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in $\lambda > LS(\mathcal{K})$ then it is $\mu$-galois-stable for each cardinal $\mu \in [LS(\mathcal{K}), \lambda)$. 
Beyond syntax?

Back to syntax!

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BEYOND THE PRESENTATION THEOREM

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A strange, strange logic
**The logic $L^1_{\kappa}$**

There is a logic called $L^1_{\kappa}$ (Shelah, 2007) in between $L_{\kappa,\omega}$ and $L_{\kappa,\kappa}$ ($\kappa$ singular strong limit):

$$L_{\kappa,\omega} \subset L^1_{\kappa} \subset L_{\kappa,\kappa}$$

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that has many desirable properties:

- Undefinability of well-order (very weak compactness)
- Interpolation (it “balances” the interpolation problem between $L_{\kappa,\omega}$ and $L_{\kappa,\kappa}$):
  if $\phi \rightarrow \psi \in L_{\kappa,\omega}$, $\phi$ has vocabulary $L_1$, $\psi$ has vocabulary $L_2$ then there is $\theta$ in the common vocabulary $L_1 \cap L_2$ such that $\phi \vdash \theta \vdash \psi$... BUT $\theta \in L_{\kappa,\kappa}$.
- Downward Löwenheim-Skolem
- Maximality for the previous properties (“Lindström”): any logic above $L_{\kappa,\omega}$ satisfying undefinability of well-order, occurrence below $\kappa$ (for $\kappa = \beth_\kappa$ strong limit) interpolation and LS must be $\leq L_{\kappa}^1$. 
THE CONNECTION WITH A.E.C.’S

(Work in progress, with Shelah)
For any a.e.c. $\mathcal{K}$ with $\tau = \tau_{\mathcal{K}}$, $\kappa = LST_{\mathcal{K}}$, $\lambda = \beth_2(\kappa + |\tau|)^+$ there exists $\psi_{\mathcal{K}} \in L_{\lambda^+\kappa^+}(\tau)$ such that $\mathcal{K} = Mod(\psi_{\mathcal{K}})$. 
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\( \psi_{\mathcal{K}} \) is in the same vocabulary as the class!!! (This provides some interesting return, some interesting symmetry to Kennedy’s description of a.e.c.’s in terms of Formalism Freeness!)
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$\psi_{\mathcal{K}}$ is in the same vocabulary as the class!!! (This provides some interesting return, some interesting symmetry to Kennedy’s description of a.e.c.’s in terms of Formalism Freeness!)

Moreover,

$$\psi_{\mathcal{K}} \in L_{\kappa^*}^1, \quad \mathcal{K} \approx L_{\kappa^*}^1.$$
Lull
Really, back to syntax???

There are many issues related to $L^1_{\kappa}$:

- No actual definition of the syntax (instead, a game - see Väänänen’s lecture a few minutes from now)
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- No actual definition of the syntax (instead, a game - see Väänänen’s lecture a few minutes from now)
- No Consistency Properties attached to the logic
- Only partial understanding of its power
ON THE EHRENFEUCHT-FRAÏSSÉ GAME

The syntax is really defined in terms of an Ehrenfeucht-Fraïssé “partial equivalence” game $G_{\Gamma,\theta,\alpha}(M_1, M_2)$:

- Player I chooses a sequence from $M_1$, 

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- Player II breaks the sequence into $\omega$ parts and chooses a sequence in $M_2$,
- Player I acts following the challenge from the breakup, on the FIRST piece and plays another sequence,
- Player II acts following the challenge from the breakup, on the SECOND piece and plays another sequence,
Infinite debts, finite time

Descriptions by Väänänen in terms of “Infinite debts, finite time to pay off them” of the game. The point: Playing the game, I “opens up” space for possible answers - possible functions - and “simulates” the role of the expansion by predicates from the Presentation Theorem.

Mysteries:

- a strong syntax for $L^1_\kappa$,
- info from a.e.c.’s?
¡Gracias! Thank you for your attention!