Model Theory of representations of operator algebras

Camilo Enrique Argoty Pulido

Universidad Nacional de Colombia
Facultad de Ciencias, Departamento de Matemáticas
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Camilo Enrique Argoty Pulido

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Alexander Berenstein
Andrés Villaveces

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A mis hijas, a mi madre, a mi padre, a mis amigos

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Resumen

En esta tesis se construyen las bases de una teoría de modelos de un espacio de Hilbert H con tres expansiones: H como una representación con operadores acotados de una C^*-álgebra, H expandido con un operador cerrado autoadjunto no acotado y H con una familia de operadores que forman una *-álgebra. Se trabaja en dos marcos principales: Lógica continua y Clases Elementales Abstractas Métricas (MAEC por sus siglas en inglés). Se obtienen resultados en estabilidad, axiomatizabilidad y caracterización de la no bifurcación para los casos anteriormente descritos.

Palabras clave: Espacio de Hilbert, C^*-Álgebra, Operador cerrado autadjunto no acotado, O^*-Álgebra, Teoría de modelos.

Abstract

In this thesis we build the basis of the model theory of the expansion of a Hilbert space by operators in three main cases: H with a C^*-algebra of bounded operators, H expanded with an unbounded self-adjoint operator and H a *-representation of a *-algebra. We work in two main frameworks: Continuous logic and the Metric Abstract Elementary Classes (MAECS). We get results on stability, axiomatizability and characterization of forking for these settings.

Keywords: Hilbert space, C^*-algebra, Closed Unbouned Operator, O^*-algebra, Model Theory.

Contents

1 Introduction

1.1 The model theory of operator algebras vs the model theory of Hilbert spaces

If one tries to study the model theory of operator algebras, one will find out very quickly that the results in that line are not very encouraging: In the work of [22], [23] and [24], Farah, Hart and Sherman explore the model theory (from the point of view of continuous logic) of operator algebras such as C^*-algebras and von Neumann algebras. They conclude that C^*-algebras are in general not stable. Since then, in the attempts to develop the model theory of operator algebras, it is very difficult even to get characterizations of elementary equivalence in the framework of continuous logic, and most results are restricted to limited families of algebras (for instance see [13]).
In this thesis we begin by proving results in the same direction: In Chapter 2, we prove that infinite dimensional factors have the TP2 property (see Corollary 2.0.4). This means that stability theory tools are not very useful when they are tried to be used directly on this kind of algebras.
However, not everything is so negative. Following the idea expressed by Grothendieck that the best way to know an algebra is to know its representations, we doscover a completely different world. In this world, objects are not only stable but may even be superstable; elementary equivalence is naturally characterized and model theoretic concepts like types correspond to classical notions from operator algebras, spectral theory and general functional analysis.
This can be seen as connected with some model theoretic properties of Hilbert spaces (see Chapter 15 in [21]). From now on, these properties will be known as the *Basic Hilbert Space Model Theoretic Properties* (BHSMTP):

Categoricity and stability One of the earlier results about Hilbert spaces, due to Hilbert himself, is that any Hilbert space is isometrically isomorphic to a Hilbert space of the form $\ell^2(B)$ for a suitable set B. This has as a consequence that the theory of infinite dimensional complex Hilbert spaces (IHS according to [21]) is λ-categorical for $\lambda > \aleph_0$, and therefore $\aleph_0$-stable.

Characterization of types In Hilbert spaces, the type of a vector is determined by its norm. This makes the Stone space $S_1(\emptyset)$ to be homeomorphic to the interval $[0, 1]$.

Quantifier elimination The previous characterization of types has the immediate consequence that IHS has quantifier elimination.

Non-forking coincides with orthogonality In Hilbert spaces, one can characterize non forking by the usual relation of orthogonality.

1.2 The results in this thesis

What we have observed in this thesis is that these properties can be extended to the case when we study an algebra acting on a Hilbert space H. So, we have considered three cases:

- A C^*-algebra of operators acting on H.
- A closed unbounded self-adjoint operator acting on H.
- An O^*-algebra of operators acting on H.

In every case we will get results that can be seen as in the same line as the results listed above for Hilbert spaces:

1. Let $\mathcal{A}$ be a (unital) C^*-algebra. Let $\pi : \mathcal{A} \to B(H)$ be a C^*-algebra non-degenerate homomorphism, where $B(H)$ is the algebra of bounded operators over a Hilbert space H. Let this structure be denoted by (H, π). Some of the results for this case are:
    a) Two representations of A are elementarily equivalent if and only if they are approximately unitarily equivalent (see Theorem 3.2.11).
    b) The theory of (H, π) is $\aleph_0$-categorical up to perturbations (see Theorem 3.3.8).
    c) The (incomplete) theory of the representations of $\mathcal{A}$ has a model companion that is called the generic representation of $\mathcal{A}$ (see Theorem 3.5.3 and Definition 3.5.4).
    d) Let (H, π) be a generic representation of $\mathcal{A}$. The stone space $S_1(Th(H, \pi))$ (i.e. the set of types of vectors of norm less than or equal to 1) with the logic topology is homeomorphic to the quasi-state space $Q_{\mathcal{A}}$ with the weak topology (see Theorem 3.5.5).
    e) The theory of (H, π) has quantifier elimination (see Corollary 3.4.7).
    f) Non-forking coincides with orthogonality and its properties lead to show that $Th(H, \pi)$ is superstable (see Theorem 3.7.7).
    g) Types have canonical bases (see Theorem 3.7.8).
    h) Orthogonality and domination of types can be expressed in terms of similar relations between their associated functionals.

    We will see that the first four results can be seen as the way the original categoricity expresses in this setting.

2. Let Q be an unbounded closed self-adjoint operator on a Hilbert space H. We will build a Metric Abstract Elementary Class (MAEC) associated with the structure (H, Γ_Q), denoted by $\mathcal{K}_{(H,\Gamma_Q)}$, where Γ_Q stands for the distance to the graph of the operator Γ_Q. Some of the results for this case are:

   a) The class $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-categorical up to a system of perturbations (see Theorem 4.4.5).

   b) Continuous first order elementary equivalence of structures of the type (H, Γ_Q) can be characterized using the Weyl-Von Neumann-Berg Theorem.

   c) The class $\mathcal{K}_{(H,\Gamma_Q)}$ cannot be axiomatized in continuous first order logic. Instead, the class $\mathcal{K}_{(H,\Gamma_Q)}$ can be axiomatized in continuous $L_{\omega_1,\omega}$ logic (see Theorem 4.5.11 and Remark 4.5.12).

   d) Galois types of vectors in some structure in $\mathcal{K}_{(H,\Gamma_Q)}$ are characterized in terms of spectral measures (see Theorem 4.2.21).

   e) Non-splitting in $\mathcal{K}_{(H,\Gamma_Q)}$ coincides with orthogonality and it has the same properties as non-forking for superstable first order theories (see Theorem 4.6.9).

   f) Orthogonality and domination of types can characterized in terms of similar relations between the spectral measures associated with them (see Corollary 4.7.2 and Corollary 4.7.6).

   Also, we can see in the first three results traces of the original categoricity in Hilbert spaces.

3. Let $\mathcal{A}$ be a *-algebra and let $\pi : \mathcal{A} \to \mathcal{L}(D)$ a *-representation of $\mathcal{A}$ on H. Let us denote that structure by (H, D, π). We build a Metric Abstract Elementary Class (MAEC) associated with the structure (H, D, π) which is denoted by $\mathcal{K}_{(H,D,\pi)}$. Some of the results are:

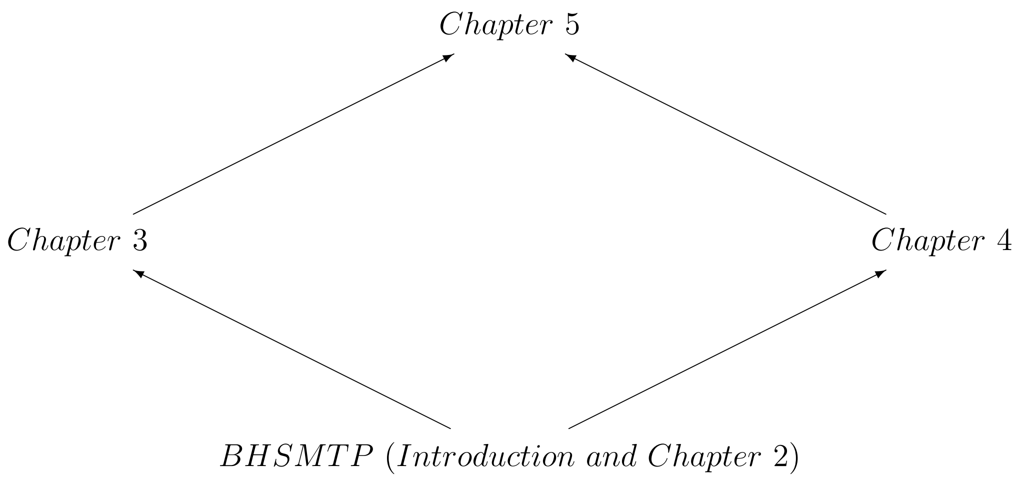
   a) Galois types of vectors can be characterized in terms of their associated functionals on the double commutant of the monster model of the class (see Theorem 5.3.4).

   b) Let $\bar{v} \in H^n$ and $E \subseteq H$. Then the (Galois) type $ga - tp(\bar{v}/E)$ has a canonical base formed by a tuple of elements in H (see Theorem 5.5.11).

   c) Non-splitting in $\mathcal{K}_{(H,D,\pi}$ coincides with orthogonality and it has the same properties as non-forking for superstable first order theories (see Theorem 5.5.3).

   d) Orthogonality and domination of types can be expressed in terms of similar relations between their associated functionals on the double commutant of the monster model of the class (see Theorem 5.6.16 and Theorem 5.6.20).

Summarizing, we can see a conceptual relationship between the different chapters in this thesis in the following diagram:



So, in a way, we can see Chapter 5 as an "amalgam"of Chapter 3 and Chapter 4 over the Basic Model Theoretic Properties expressed in the Introduction.

1.3 Types interpreted in quantum mechanics

One of the most remarkable facts in this thesis is the capability to characterize types in terms of other notions which are familiar to specialists in functional analysis and operator theory. Even more, this characterization of types can give us another interpretation of states in quantum systems: States can be interpreted exactly as Galois types. In other words, types are what we can get from quantum systems experimentally. These connections work as an illustration of new insights on quantum physics when we use logical notions such as "type", "theory", "model"and "forking independence". This corresponds to ideas from a recent paper by Briggs, Butterfield and Zellinger called: **"The Oxford Questions on the foundations of quantum physics"** (see [15]). In this paper, the authors pose several questions that arised in a conference celebrated in Oxford University in 2010. Among the questions listed by the authors we find the following that can be considered of our concern:

- What can we learn about quantum physics by using the notion of information?
- What insights are to be gained from category-theoretic, informational, geometric and operational approaches to formulating quantum theory?
- How does different aspects of the notion of reality influence our assesment of the different interpretations of quantum theory?

Since stability, independence and forking are strongly related to the notion of information, we think that the apprach given in this thesis goes in the right direction to help to solve these questions.

1.4 Further work

As most works, this thesis leaves more questions open than not. The following topics arise from what we have seen so far:

1. Investigating the Zariski-like geometries underlying the decomposition of representations of C-algebras.
2. Proving a Voiculescu-like theorem for -algebra representations
3. Developing stronger tools to deal with the model theory of C-algebras themselves. Some possibilities may be accessible categories and model theoretic treatment of Hilbert modules, similar to the one developed in this work.

1.5 Previous work related to this thesis

This work continues the works done by the author of this thesis in [4]. There, we study the model theory of a Hilbert space H expanded either with an unitary operator U or with a normal operator N on H.
There are many works related to this thesis, and they can be classified in two groups. The first ones deal with the model theory of Hilbert spaces expanded with some operators in the framework of continuous logic. The second ones focus on generalizing the notion of Abstract Elementary Class (see [33]), to the setting metric structures along with its further purely model theoretical analysis.
The work on model theory fo the expansions of Hilbert spaces by operators goes back to José Iovino's Ph.D. thesis (see [25]), where he and C. W. Henson noticed that the structure $(H, 0, +, \langle|\rangle, A)$, where A is a bounded operator, is stable. In [7], Alexander Berenstein and Steven Buechler gave a geometric characterization of forking in those structures, when the operator is unitary, after adding to it the projections determined by the Spectral Decomposition Theorem. In [20] Itaï Ben Yaacov, Alexander Usvyatsov and Moshe Zadka characterized the unitary operators corresponding to generic automorphisms of a Hilbert space as those unitary transformations whose spectrum is the unit circle, and gave the key ideas used in this thesis to characterize domination and orthogonality of types. Argoty and Berenstein (see [4]) studied the theory of the structure $(H, +, 0, \langle|\rangle, U)$ where U is a unitary operator in the case when the spectrum is countable. Finally, the author and Ben Yaacov (see [5]) studied the case of a Hilbert space expanded by a normal operator N.
On the other hand, in the 1980's Saharon Shelah defined in [33] the notion of a so called *Abstract Elementary Class* (AEC) as a generalization of the notion of elementary class, which is a class of models of a first order theory. This paper from Shelah generated a new trend in model theory toward the study of these classes. In order to deal with the case of metric structures, Tapani Hyttinen and Åsa Hirvonen defined in [18, 19] the notion of *Metric*

Abstract Elementary Classes (MAEC). After this, in [35, 34, 38] Andrés Villaveces and Pedro Zambrano studied notions of independence and superstability for MAEC's. With respect to the work of the author itself: Since [4], the author's work has evolved towards relating tools from representation theory of some classes of operator algebras with model theoretic notions. The author has worked on unitary operators, normal operators, abelian C^*-algebras or operators, arbitrary C^*-algebras of operators, closed unbounded selfadjoint operators, -representations and, finally, the most recent work of the author goes towards dealing with the model theory of operator algebras themselves instead of their representations. Most of results in Chapter 3 have been published in [1] and [3]. In the same way, most results in Chapter 4 have been published in [3]. The results in Chapter 2 and Chapter 5 are unpublished.

1.6 Contents of this thesis

This thesis is divided as follows: In Chapter 2, we study directly some model theoretic propeties of C^*-algebras, the main result is that many important C^*-algebras have the TP2 property. In Chapter 3, we work on the model theory of a representation of a C^*-algebra. In Chapter 4, we deal with a Hilbert space H expanded with an unbounded self-adjoint operator. Finally, in Chapter 5, we work on a -representation of -algebra. We will assume that the reader is familiar with basic concepts of spectral theory, for example the material found in [31] among others. We will introduce technical results as we need them.

2 Motivation: Operator algebras do not have nice model theoretic properties

It had been suspected long ago that C^*-algebras are not well behaved in the model theoretic sense. This has been stated since C. W. Henson and J. Iovino. More recently, Farah et al (see [22], [23] and [24]) have shown that in general C^*-algebras are not stable. In this section we go further: we show that in fact, operator algebras not only are not stable but they have the independence property or even the TP2 property.
Roughly speaking, this hapens because in general we can subdivide projections in a C^*-algebra at will and it is possible to build witnesses for the order and tree properties with these projections. This is the way we prove the main results of this chapter which are the following:

1. The algebra $\mathcal{K}(H)$ of compact operators on an infinite dimensional Hilbert space H has the independence property (see Corollary 2.0.2).

2. Factors of types I_∞, II and III have the TP2 property (see Corollary 2.0.4).

The preliminaries about C^* and von Neumann algebras that are needed to understand the results in this chapter can be found in Section .2 of the Appendix, mainly the notions of compact operator, factor and the type in which a factor can be classified.
Take $\mathcal{A}$ to be a unital C^*-algebra (see Definition .2.1 and Definition .2.8). The continuous first order structure for $\mathcal{A}$ is:

$$(\mathcal{A}, 0, e, (f_{\alpha,\beta})_{(\alpha,\beta)\in Ball_1(\mathbb{C})}, \cdot, {}^*, \|\cdot\|)$$

The next theorem comes from an unpublished result from Alexander Berenstein in the sense that the algebra $\mathcal{K}(H)$ of compact operators on an infinite dimensional Hilbert space H has the independence property. In this case we isolate the main argument and show that it can be used for many other cases as are the infinite dimensional factors:

Theorem 2.0.1. *Any C^*-algebra such that for any $n \in \mathbb{N}^+$ there are n disjoint projections (see Definition .2.19), has the independence property.*

Proof. Let S a subset of $\{1, \dots, n\}$. Let $(p_i)_{i=1,\dots,n}$ be disjoint projections. Let p_S be the projection

$$p_S := \bigvee_{i\in S} p_i$$

Then, for $k \in \{1, \dots, n\}$ we have that $p_k \cdot p_S = 0$ if and only if $k \notin S$. So, the condition $\phi : \|pq\| = 0$ and the pair of sequences $(p_k)_{k=1,\dots,n}$ and $(p_S)_{S \subseteq \{1,\dots,n\}}$ witness the independence property. ☐

Using this, we can now generalize Berenstein's observation:

Corollary 2.0.2. *The following algebras have the independence property:*

1. *The algebra $\mathcal{K}(H)$ of compact operators on an infinite dimensional Hilbert space H (see Definition .2.10.*
2. *Factors of types I_∞, II and III (see Definition .2.22 and Defintion .2.26).*

Proof. 1. Finite rank projections are compact. Since the Hilbert space is infinite dimensional, there are infinite disjoint projections in $\mathcal{K}(H)$.

2. By Theorem .2.32, the dimension of minimal projections in factors of type I_∞ and type II have infinite rank ($\{1, \dots, \infty\}$ for factors of type I_∞, $[0, 1]$ for factors of type II_1 and $[0, \infty]$ of type II_∞). This means that we have n disjoint projections for every $n \in \mathbb{Z}^+$. So we can apply Theorem 2.0.1. For the case of type III, recall that in these factors there is no finite projection. So, for every projection P, there is a projection $P' < P$ such that $P' \sim P$. Note that $P - P' < P$ and we get two disjoint projection less than P. Then, if we start with the identity, we get 2^n disjoint projections in n steps. By Theorem 2.0.1, we get the independence property.

☐

However, immediately comes the question on how further this argument can go. Since factors have interesting properties (first order describable) about their projections it will be worthy to generalize previous argument to proof not only independence property but TP2 property:

Theorem 2.0.3. *Any C^*-algebra such that for any $n \in \mathbb{N}^+$ every projection has at least n disjoint proper subprojections, has the $TP2$.*

Proof. Since I is a projection, it has n disjoint subprojections. So, there are n disjoint projections $P_1, \dots, P_n$. Let $s \in n^{\leq n}$ and let P_s be a projection already defined. Let $P_{s^\frown 1}, P_{s^\frown 2}, \dots P_{s^\frown n}$ be disjoint subprojections of P_s. Define inductively a_{mn} in the following way:

- $a_{0j} := P_j$ for $j = 1, \cdots, n$.
- $a_{ij} := \bigvee_{s \in i^n} P_{s^\frown j}$

Let $\phi(X, a_{mn})$ be the condition $\|a_{mn} X\| = 1$. Let $(n_k)_{k=1,\cdots,n}$ be a finite sequence in $\mathbb{N} \cap [0, n]$. Note that if $i_1 < i_2$, then for every k_1 and k_2 $a_{i_1 k_1} a_{i_2 k_2} \neq 0$. Then the set

$\{\phi(X, a_{i n_i}) \mid i = 1, \cdots, n\}$

is consistent since $\Pi_{i=1}^{n} a_{in_i} \neq 0$. On the other hand, for k_1, $k_2 = 1, \cdots, n$ and any $i = 1, \cdots, n$ $a_{ik_1} a_{ik_2} = 0$. So, we have build a $n \times n$ matrix of conditions such that all rows are mutually inconsistent and every path between rows is consistent. By compactness, the algebra has the TP2. □

Once again, previous theorem can be used to prove TP2 in particular cases:

Corollary 2.0.4. *Factors of types I_∞, II and III have the $TP2$*

Proof. **Factors of type** I_∞ By Theorem .2.30, a factor of type I_∞ is isometrically isomorphic to $B(\ell^2)$. So, we will show that $B(\ell^2)$ has the TP2. Let P_n be the operator that takes a sequence $(x_1, x_2, \dots)$ to the sequence that has zero in all components, except the ones corresponding to the powers of the n^{th} prime number. Then we get infinite disjoint projections. In the same way, for an element in the range of P_n, apply the operators that take a sequence $(x_1, x_2, \dots)$ to the sequence that has zero in all components, except the ones corresponding to the powers of the n^{th} prime number, whose exponent is itself a power of another prime number. So for any projection built so far, we can get infinite non-trivial disjoint subprojections. Therefore, by Theorem 2.0.3, $B(\ell^2)$ has the TP2.

Factors of type II **and** III Every projection can be halved. So for every projection, we can build 2^n subprojections for all $n \in \mathbb{N}$. By Theorem 2.0.3, these factors have the TP2.

□

This gives much evidence that usual model theoretic tools are useless for dealing with C^*-algebras. So, we change the perspective and we study the model theory of the representations of a C^*-algebra to see more positive results.

3 The model theory of modules of a C^*-algebra

3.1 Introduction

Let $\mathcal{A}$ be a (unital) C^*-algebra (see Section .3) and let $\pi : \mathcal{A} \to B(H)$ be a C^*-algebra non-degenerate isometric homomorphism, where $B(H)$ is the algebra of bounded operators over a Hilbert space H. The goal of this chapter is to study H as a metric structure expanded by $\mathcal{A}$ from the point of view of continuous logic (see [21] and [36]).
In this chapter we get the following results:

1. Furthermore, for $v, w \in H$, $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $\phi_v = \phi_w$. Here, $\phi_v : \mathcal{A} \to \mathbb{C}$ is the positive linear functional defined by the formula $\phi_v(a) := \langle av \mid v \rangle$ (see Theorem 3.4.6) .
2. The theory of (H, π) has quantifier elimination (see Corollary 3.4.7).
3. An explicit description of the model companion of $Th(H, \pi)$ (see Lemma 3.5.3).
4. A characterization of non-forking $Th(H, \pi)$ (see Theorem 3.7.7).
5. The theory $Th(H, \pi)$ is superstable (see Theorem 3.7.7).
6. Let $\bar{v} \in H^n$ and $E \subseteq H$. Then the type $tp(\bar{v}/E)$ has a canonical base formed by a tuple of elements in H (see Theorem 3.7.8).
7. Let $E \subseteq H$, p, $q \in S_1(E)$ be stationary and v, $w \in H$ be such that $v \models p$and $w \models q$. Then $p \perp_E q$ if and only if $\phi_{P^{\perp}_{acl(E)}(v_e)} \perp \phi_{P^{\perp}_{acl(E)}(w_e)}$ (see Theorem 3.8.8).
8. Let $E \subseteq H$, p, $q \in S_1(E)$ be stationary and v, $w \in H$ be such that $v \models p$and $w \models q$. Then $p \rhd_G q$ if and only if there exist v, $w \in \tilde{H}$ such that $tp(v/G)$ is a non-forking extension of p, $tp(w/G)$ is a non-forking extension of q and $\phi_{P^{\perp}_{acl(G)} w_e} \leq \phi_{P^{\perp}_{acl(G)} v_e}$. (see Theorem 3.8.12).

This chapter is divided as follows: In Section 3.2, we give an explicit axiomatization of $Th(H, \pi)$; many of the results from this section were proved by C.W. Henson but did not appear in print. In Section 3.3 we give a description of the models of $Th(H, \pi)$ and build the

monster model for the theory. In Section 3.4, we characterize the types over the emtpy set as positive linear functionals on $\mathcal{A}$ and prove quantifier elimination. Finally, in section 3.5, we build a model companion for the incomplete theory of all non-degenerate representations of a C^*-algebra $\mathcal{A}$.
The theoretical preliminaries about C^*-algebras and their representations, needed in the rest of the chapter are in Section .3. We will assume the reader is familiar with basic concepts of spectral theory, for example the material found in [31]. We will recall technical results from [31] as we need them.

3.2 The theory $IHS_{\mathcal{A},\pi}$

In this section we provide an explicit axiomatization of $Th(H, \pi)$. The main tool here is Theorem .3.16 which is mainly a consecuence of Voiculescu's theorem (see [14]). This Theorem states that two separable representations (H_1, π_1) and (H_2, π_2) of a separable C^*-algebra $\mathcal{A}$ are approximately unitarily equivalent if and only if for every $a \in \mathcal{A}$, rank($\pi_1(a)$) =rank($\pi_2(a)$). This last statement can be expressed in continuous first order logic and is the first step to build the axiomatization we mentioned above. Lemma 3.2.10, Theorem 3.2.11 and Corollary 3.2.21 are remarks and unpublished results from C. Ward Henson.
In order to describe the structure of H as a module for $\mathcal{A}$ (see Definition .3.1), we include a symbol $\dot{a}$ in the language of the Hilbert space structure whose interpretation in H will be $\pi(a)$ for every a in the unit ball of $\mathcal{A}$. Following [36], we study the theory of H as a metric structure of only one sort:

$$(Ball_1(H), 0, -, i, \tfrac{x+y}{2}, \|\cdot\|, (\pi(a))_{a\in Ball_1(\mathcal{A})})$$

where $Ball_1(H)$ and $Ball_1(\mathcal{A})$ are the corresponding unit balls in H and $\mathcal{A}$ respectively; 0 is the zero vector in H; $- : Ball_1(H) \to Ball_1(H)$ is the function that to any vector $v \in Ball_1(H)$ assigns the vector $-v$; $i : Ball_1(H) \to Ball_1(H)$ is the function that to any vector $v \in Ball_1(H)$ assigns the vector iv where $i^2 = -1$; $\frac{x+y}{2} : Ball_1(H) \times Ball_1(H) \to Ball_1(H)$ is the function that to a couple of vectors v, $w \in Ball_1(H)$ assigns the vector $\frac{v+w}{2}$; $\|\cdot\| : Ball_1(H) \to [0, 1]$ is the norm function; $\mathcal{A}$ is an unital C^*-algebra; $\pi : \mathcal{A} \to B(H)$ is a C^*-algebra isometric homomorphism. The metric is given by $d(v, w) = \|\frac{v-w}{2}\|$. Briefly, the structure will be refered to as (H, π).
It is worthy noting that with this language, we can define the inner product taking into account that for every v, $w \in Ball_1(H)$,

$$\langle v \mid w\rangle = \|\tfrac{v+w}{2}\|^2 - \|\tfrac{v-w}{2}\|^2 + i(\|\tfrac{v+iw}{2}\|^2 - \|\tfrac{v-iw}{2}\|^2)$$

Because of this reason, we will make free use of the inner product as if it were included in the language. In most arguments, we will forget this formal point of view, and will treat H directly. To know more about the continuous logic point of view of Banach spaces please see [36], Section 2.

Definition 3.2.1. Let $IHS_{\mathcal{A}}$ be the theory of Hilbert spaces together with the following conditions:

1. For $v \in Ball_1(H)$ and a, $b \in Ball_1(\mathcal{A})$:

   $\dot{(ab)}v = (\dot{a}\dot{b})v = \dot{a}(\dot{b}v)$

2. For $v \in Ball_1(H)$ and a, $b \in Ball_1(\mathcal{A})$:

   $\dot{(\frac{a+b}{2})}(v) = \frac{\dot{a}+\dot{b}}{2}(v) = \frac{\dot{a}v+\dot{b}v}{2}$

3. For v, $w \in Ball_1(H)$, and $a \in Ball_1(\mathcal{A})$:

   $\dot{a}\left(\frac{v+w}{2}\right) = \frac{\dot{a}v+\dot{a}w}{2}$

4. For $v \in Ball_1(H)$ and $a \in Ball_1(\mathcal{A})$:

   $\langle \dot{a}v \mid w\rangle = \langle v \mid \dot{a^*}w\rangle$

5. a) For $a \in Ball_1(\mathcal{A})$:

      $\sup_v(\|\dot{a}v\| \dot{-} \|a\|\|v\|) = 0$

   b) For $a \in Ball_1(\mathcal{A})$:

      $\inf_v \max(|\|v\| - 1|, |\|\dot{a}v\| - \|a\| \,|) = 0$

6. For $v \in Ball_1(H)$ and e the identity element in $\mathcal{A}$:

   $\dot{(ie)}v = iv$

   $\dot{e}v = v$

Remark 3.2.2. By Fact .3.3, Item (6) implies that the representation is non-degenerate. Therefore, $IHS_{\mathcal{A}}$ is the theory of the non-degenerate representations of a fixed C^*-algebra $\mathcal{A}$. Conditions in Item (5), are natural continuous logic conditions that say that $\|\pi(a)\| = \|a\|$.

Remark 3.2.3. Since the rationals of the form $\frac{k}{2^n}$ are dense in $\mathbb{R}$, Item (3) and Item (6) are enough to show that for all $v \in Ball_1(H)$, all $a \in Ball_1(\mathcal{A})$ and all $\lambda \in \mathbb{C}$, we have that $\dot{(\lambda a)}v = \lambda(\dot{a}v)$.

Lemma 3.2.4. *If $S : H \to H$ is a bounded operator, S is non-compact if and only if for some $\lambda_S > 0$, $S(Ball_1(H))$ contains an isometric copy of the ball of radius λ_S of ℓ^2 i.e., there exists an orthonormal sequence $(w_i)_{i\in\mathbb{N}} \subseteq S(Ball_1(H))$ and a vector sequence $(u_i)_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that for every $i \in \mathbb{N}$, $Su_i = \lambda_S w_i$.*

Proof. Suppose S is non-compact. Then there is a sequence $(u_i')_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that no subsequence of $(Su_i')_{i\in\mathbb{N}}$ is convergent. Since the unit ball in $\mathbb{C}^n$ is compact, $dim(span(Su_i')_{i\in\mathbb{N}}) = \infty$. By the Grahm-Schmidt process we can assume that $(Su_i')_{i\in\mathbb{N}}$ is an orthogonal sequence. Since no subsequence of $(Su_i')_{i\in\mathbb{N}}$ converges, we have that $\liminf\{\|Su_i'\| \mid i \in \mathbb{N}\} > 0$ (otherwise there would be a subsequence of Su_i' converging to 0). Let $\lambda_S := \frac{\liminf\{\|Su_i'\| \mid i\in\mathbb{N}\}}{2} > 0$. For $i \in \mathbb{N}$, let $u_i := \frac{\lambda_S u_i'}{\|Su_i'\|}$ and $w_i := \frac{Su_i'}{\|Su_i'\|}$. Without loss of generality, we can asume that for all $i \in \mathbb{N}$, $\|Su_i'\| > \lambda_S$ and therefore $\|u_i\| \leq 1$. Then, $Su_i = S(\frac{\lambda_S u_i}{\|Su_i\|}) = \lambda_S \frac{Su_i'}{\|Su_i'\|} = \lambda_S w_i$.
On the other hand, suppose there are $\lambda_S > 0$, an orthonormal sequence $(w_i)_{i\in\mathbb{N}} \subseteq S(Ball_1(H))$ and a vector sequence $(u_i)_{i\in\mathbb{N}} \subseteq Ball_1(H)$ such that for every $i \in \mathbb{N}$, $Su_i = \lambda_S w_i$. Then no subsequence of $(Su_i)_{i\in\mathbb{N}}$ converges and S is non-compact. □

Remark 3.2.5. If in Lemma 3.2.4 $\|S\| \leq 1$, it is clear that $\lambda_S \leq 1$.

Lemma 3.2.6. *Let $a \in Ball_1(\mathcal{A})$ be such that $\pi(a)$ is a non-compact operator on H. Let $\lambda_{\pi(a)}$, $(u_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ be as described in Lemma 3.2.4. Then, for every $n \in \mathbb{N}$*

$$(H,\pi) \models \inf_{u_1,u_2\cdots u_n} \inf_{w_1,w_2\cdots w_n} \max_{i,j=1,\cdots,n} \big(|\langle w_i \mid w_j\rangle - \delta_{ij}|, |au_i - \lambda_{\pi(a)}w_i|\big) = 0 \quad (3\text{-}1)$$

Proof. This condition is a continuous logic condition for:

$$\exists u_1u_2\cdots u_n \exists w_1w_2\cdots w_n \wedge \big(\bigwedge_{i,j=1,\cdots,n} \langle w_i \mid w_j\rangle = \delta_{ij}\big) \wedge \big(\bigwedge_{i=1,\cdots,n} \dot{a}u_i = \lambda_{\pi(a)}w_i\big)$$

where δ_{ij} is Kronecker's delta. By Lemma 3.2.4, this set of conditions says that $\pi(a)(Ball_1(H))$ contains an isometric copy the ball of radius $\lambda_{\pi(a)}$ of ℓ^2. □

Remark 3.2.7. It is an easy consecuence of Riesz representation theorem that if $S : H \to H$ is an operator with rank n, then there exist two orthonormal families $E_1 := \{u_1, \cdots, v_n\}$, $E_2 := \{w_1, \cdots, w_n\}$ and a family $\{\alpha_i, \ldots, \alpha_n\}$ of non-zero complex numbers such that for every $v \in H$, $Sv = \sum_{i=1}^n \alpha_i\langle v \mid u_i\rangle w_i$. Furthermore, if R is a compact operator, there is a complex sequence $(\alpha_i)_{i\in\mathbb{N}^+}$ that converges to 0 such that for every $v \in H$, $Rv = \sum_{i=1}^\infty \alpha_i\langle v \mid u_i\rangle w_i$. If $\|R\| \leq 1$, then for every i, $|\alpha_i| \leq 1$.

Lemma 3.2.8. *Let $n \in \mathbb{N}$ and $a \in Ball_1(\mathcal{A})$ be such that $rank(\pi(a)) = n$. Let $\{\alpha_i, \ldots, \alpha_n\}$ complex numbers as described in 3.2.7. Then*

$$(H,\pi) \models \inf_{u_1u_2\cdots u_n} \inf_{w_1w_2\cdots w_n} \sup_v \max_{i,j=1\cdots n} \big(|\langle u_i \mid u_j\rangle - \delta_{ij}|, |\langle w_i \mid w_j\rangle - \delta_{ij}|,$$
$$, \|\dot{a}v - \sum_{k=1}^n \alpha_i\langle v \mid u_i\rangle w_i)\|\big) = 0 \quad (3\text{-}2)$$

Proof. This condition is a continuous logic condition for:

$$\exists u_1 u_2 \cdots u_n \exists w_1 w_2 \cdots w_n \big(\bigwedge_{i,j=1,\cdots,n} \langle u_i \mid u_j \rangle = \delta_{ij} \wedge \langle w_i \mid w_j \rangle = \delta_{ij} \big) \wedge$$

$$\wedge \forall v (\dot{a} v = \sum_{k=1}^{n} \alpha_i \langle v \mid u_i \rangle w_i) \quad (3\text{-}3)$$

where δ_{ij} is Kronecker's delta. □

Remark 3.2.9. If Condition (3-2) is valid for some $a \in \mathcal{A}$, and some tuple $\{\alpha_1, \ldots, \alpha_n\}$, by Remark 3.2.7 it is clear that $\pi(a)$ has rank n.

Recall that all C^*-algebras under consideration are unital and all representations are nondegenerate. However, in the next lemma we do not use the hypothesis that $\mathcal{A}$ is unital.

Lemma 3.2.10. *Let $\mathcal{A}$ be a separable C^*-algebra of operators on the separable Hilbert space H, and π_1 and π_2 two non-degenerate representations of $\mathcal{A}$ on H. Then the structures (H, π_1) and (H, π_2) are elementarily equivalent if and only if π_1 and π_2 are approximately unitarily equivalent.*

Proof. $\Rightarrow$ Suppose $(H, \pi_1) \equiv (H, \pi_2)$. Let $a \in Ball_1(\mathcal{A})$ and assume that $\text{rank}(\pi_1(a)) = n < \infty$ then Condition (3-2) will hold in (H, π_1). By elementary equivalence, Condition (3-2) will hold in (H, π_2) and therefore $\text{rank}(\pi_2(a)) = n$. In the same way, if $\text{rank}(\pi_1(a)) = \infty$, Condition (3-1) will hold in the structure (H, π_1) for every n. By elementary equivalence, Condition (3-1) will hold for every $a \in \mathcal{A}$ and every n in the structure (H, π_2) and $\text{rank}(\pi_2(a)) = \infty$. This implies that the hypotesis of Theorem .3.16 hold, and therefore π_1 and π_2 are approximately unitarily equivalent.

$\Leftarrow$ Suppose π_1 and π_2 are approximately unitarily equivalent. Then, there exists a sequence of unitary operators $(U_n)_{n<\omega}$ such that for every $a \in \mathcal{A}$, $\pi_2(a) = \lim_{n\to\infty} U_n \pi_1(a) U_n^*$. Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$. Then $\Pi_{\mathcal{F}}(H, U_n \pi_1(A) U_n^*) = \Pi_{\mathcal{F}}(H, \pi_2)$. On the other hand, since for every n, $(H, U_n \pi_1(A) U_n^*) \simeq (H, \pi_1)$, then $\Pi_{\mathcal{F}}(H, U_n \pi_1(A) U_n^*) \simeq \Pi_{\mathcal{F}}(H, \pi_1)$. So, $\Pi_{\mathcal{F}}(H, \pi_1) \simeq \Pi_{\mathcal{F}}(H, \pi_2)$ and therefore $(H, \pi_1) \equiv (H, \pi_2)$.

□

Theorem 3.2.11. *Let $\mathcal{A}$ be a C^*-algebra, H_1 and H_2 be Hilbert spaces, and π_1 and π_2 be two representations of $\mathcal{A}$ on H_1 and H_2 respectively. Then the structures (H_1, π_1) and (H_2, π_2) are elementarily equivalent if and only if for all $a \in \mathcal{A}$, rank$(\pi_1(a))$ = rank$(\pi_2(a))$.*

Proof. $\Rightarrow$ Suppose (H_1, π_1) and (H_2, π_2) are elementarily equivalent and let $a \in \mathcal{A}$. By Theorem 3.2.6 and Theorem 3.2.8, $\text{rank}(\pi(a)) = n$ and $\text{rank}(\pi(a)) \geq n$ are sets of conditions in $L(\mathcal{A})$. By elementary equivalence, $\text{rank}(\pi_1(a)) = \text{rank}(\pi_2(a))$.

$\Leftarrow$ Let (H_1,π_1) and (H_2,π_2) be such that $\mathrm{rank}(\pi_1(a)) = \mathrm{rank}(\pi_2(a))$ for all $a \in \mathcal{A}$, and let $\phi(a_1,\cdots,a_n) = 0$ be a condition in $L(\mathcal{A})$. Let $\hat{\mathcal{A}} \subseteq \mathcal{A}$ be the unital sub C^*-algebra of $\mathcal{A}$ generated by $\bar{a} = (a_1,\cdots,a_n)$, and $\hat{\pi}_1$ and $\hat{\pi}_2$ be the restrictions of π_1 and π_2 to $\hat{\mathcal{A}}$ (note that $\hat{\pi}_1(e) = I = \hat{\pi}_1(e)$). Then $\hat{\mathcal{A}}$ is separable and by Löwenheim-Skolem Theorem and Fact .3.3, there are two separable non-degenerate representations $(\tilde{H}_1,\tilde{\pi}_1)$ and $(\tilde{H}_2,\tilde{\pi}_2)$ of $\hat{\mathcal{A}}$ which are elementary substructures of $(H_1,\hat{\pi}_1)$ and $(H_2,\hat{\pi}_2)$ respectively. By Theorem .3.16 $(\tilde{H}_1,\tilde{\pi}_1)$ is approximately unitarily equivalent to $(\tilde{H}_2,\tilde{\pi}_2)$. By the previous lemma, $(\tilde{H}_1,\tilde{\pi}_1)$ and $(\tilde{H}_2,\tilde{\pi}_2)$ are elementary equivalent.

Then, $(\hat{H}_1,\hat{\pi}_1) \models \phi(a_1,\cdots,a_n) = 0$ if and only if $(\hat{H}_2,\hat{\pi}_2) \models \phi(a_1,\cdots,a_n) = 0$. But $(H,\pi_1) \models \phi(a_1,\cdots,a_n) = 0$ if and only if $(\hat{H}_1,\hat{\pi}_1) \models \phi(a_1,\cdots,a_n) = 0$ and $(H,\pi_2) \models \phi(a_1,\cdots,a_n) = 0$ if and only if $(\hat{H}_2,\hat{\pi}_2) \models \phi(a_1,\cdots,a_n) = 0$. Then $(H,\pi_1) \models \phi(a_1,\cdots,a_n) = 0$ if and only if $(H,\pi_2) \models \phi(a_1,\cdots,a_n) = 0$.

□

Notation 3.2.12. For a Hilbert space H and a positive integer n, $H^{(n)}$ denotes the direct sum of n copies of H. If $S \in B(H)$, $S^{(n)}$ denotes the operator on $H^{(n)}$ given by $S^n(v_1,\cdots,v_n) = (Sv_1,\cdots,Sv_n)$. If $\mathcal{B} \subseteq B(H)$, $\mathcal{B}^{(n)}$ is the set $\{S^{(n)} \mid S \in \mathcal{B}\}$.

Definition 3.2.13. A representation (H,π) of $\mathcal{A}$ ($\mathcal{A}$ not necessarily unital) is called *compact* if $\pi(\mathcal{A}) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the algebra of compact operators on H.

Remark 3.2.14. In case that $ker(\mathcal{A}) = 0$, ($\mathcal{A}$ no necessarilly unital) we have that this representation is non-degenerate.

Remark 3.2.15. Recall that if $R \in \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i)^{(k_i)}$, then there is a sequence $(R_i)_{i\in\mathbb{Z}^+}$ such that $R_i \in \mathcal{K}(H_i)^{(k_i)}$ and $R = \sum_{i\in\mathbb{Z}^+} R_i$ in the norm topology. This means, in particular, that $\lim_{i\to\infty} \|R_i\| = 0$.

Definition 3.2.16. Let (H,π) be a representation of $\mathcal{A}$. We define:

The essential part of π It is the C^*-algebra homomorphism,

$$\pi_e := \rho \circ \pi : \mathcal{A} \to B(H)/\mathcal{K}(H)$$

of $\pi(\mathcal{A})$, where ρ is the canonical proyection of $B(H)$ onto the Calkin Algebra $B(H)/\mathcal{K}(H)$.

The discrete part of π It is the restriction,

$$\pi_d : ker(\pi_e) \to \mathcal{K}(H)$$
$$a \to \pi(a)$$

The discrete part of $\pi(\mathcal{A})$ It is defined in the following way:

$$\pi(\mathcal{A})_d := \pi(\mathcal{A}) \cap \mathcal{K}(H).$$

The essential part of $\pi(\mathcal{A})$ It is the image $\pi(\mathcal{A})_e$ of $\pi(\mathcal{A})$ in the Calkin Algebra.

The essential part of H It is defined in the following way:

$$H_e := \ker(\pi(\mathcal{A})_d)$$

The discrete part of H It is defined in the following way:

$$H_d := \ker(\pi(\mathcal{A})_d)^{\perp}$$

The essential part of a vector $v \in H$ It is the projection v_e of v over H_e.

The discrete part of a vector $v \in H$ It is the projection v_d of v over H_d.

The essential part of a set $E \subseteq H$ It is the set

$$E_e := \{v_e \mid v \in E\}$$

The discrete part of a set $G \subseteq H$ It is the set

$$E_d := \{v_d \mid v \in G\}$$

Lemma 3.2.17. *Let* (H_1, π_1) *and* (H_2, π_2) *be two non-degenerate representations of* $\mathcal{A}$*. If* $(H_1, \pi_1) \equiv (H_2, \pi_2)$ *then* $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$*.*

Proof. For a given representation π, let $\pi(\mathcal{A})_f$ be the (not necessarilly closed) algebra of finite rank operators in $\pi(\mathcal{A})$. If $(H_1, \pi_1) \equiv (H_2, \pi_2)$, by Lemma 3.2.8, $\pi_1(\mathcal{A})_f \simeq \pi_2(\mathcal{A})_f$ and by density of $\pi(\mathcal{A})_f$ in $\pi(\mathcal{A})_d$, we have that $\pi_1(\mathcal{A})_d \simeq \pi_2(\mathcal{A})_d$. Let $\mathcal{B} := \pi_1(\mathcal{A})_d \simeq \pi_2(\mathcal{A})_d$. Since $(H_1)_d$ and $(H_2)_d$ are the orthogonal complements of $ker(\mathcal{B})$ in H_1 and H_2 respectively, we get that $\big((H_1)_d, (\pi_1)_d\big)$ and $\big((H_2)_d, (\pi_2)_d\big)$ are non-degenerate representations of $\mathcal{B}$. Then by Theorem .3.20, $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$. ☐

Remark 3.2.18. For $E \subseteq H$, $(H_E)_e = H_{E_e}$ and $(H_E)_d = H_{E_d}$

Definition 3.2.19. Let $IHS_{\mathcal{A},\pi}$ be the theory $IHS_{\mathcal{A}}$ with the following aditional conditions:

1. For $a \in Ball_1(\mathcal{A})$ such that $\pi(a)$ is a non-compact operator on H, let $\lambda_{\pi(a)}$, $(u_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ be as described in Lemma 3.2.4. For $n \in \mathbb{N}$

$$\inf_{u_1,u_2\cdots u_n} \inf_{w_1,w_2\cdots w_n} \max_{i,j=1,\cdots,n} \big(|\langle w_i \mid w_j\rangle - \delta_{ij}|, |au_i - \lambda_{\pi(a)} w_i|\big) = 0$$

2. For $a \in Ball_1(\mathcal{A})$, such that $\text{rank}(\pi(a)) = n \in \mathbb{N}$. Let $\alpha_1, \cdots, \alpha_n$ be complex number as described in Remark 3.2.7.

$$\inf_{u_1 u_2\cdots u_n} \inf_{w_1 w_2\cdots w_n} \max\{ \max_{i,j=1,\cdots,n} \big(|\langle w_i \mid w_j\rangle - \delta_{ij}|\big), \sup_v (|\dot{a}v - \sum_{k=1}^{n} \alpha_i \langle v \mid u_i\rangle w_i)|\} = 0$$

Remark 3.2.20. We gave in Lemmas 3.2.6 and 3.2.8 the complete continuous logic formalism only for these two conditions. We omit an explicit condition describing compact infinite rank operators in $\pi(\mathcal{A})$ because they completely determined by the finite rank operators in $\pi(\mathcal{A})$.

Corollary 3.2.21. *$IHS_{\mathcal{A},\pi}$ axiomatizes the theory $Th(H,\pi)$.*

Proof. By Theorem 3.2.11. □

Remark 3.2.22. Since every model of $IHS_{\mathcal{A}}$ is a non-degenerate representation (H,π), previous corollary shows that the completions of $IHS_{\mathcal{A}}$ are of the form $IHS_{\mathcal{A},\pi}$ for some π.

3.3 The models of $IHS_{\mathcal{A},\pi}$

In this section we provide an explicit description of the models of the theory $IHS_{\mathcal{A},\pi}$. This description can be summarized by stating that every model of $IHS_{\mathcal{A},\pi}$ can be decomposed into an algebraic part (the *discrete part*) and a non algebraic part (the *essential part*). Finally we get an explicit description of the monster model of $IHS_{\mathcal{A},\pi}$ (Theorem 3.3.6).

Definition 3.3.1. Given $E \subseteq H$ and $v \in H$, we denote by:

1. H_E, the Hilbert subspace of H generated by the elements $\pi(a)v$, where $v \in E$ and $a \in \mathcal{A}$.
2. $\pi_E := \{\pi(a) \restriction H_E \mid a \in \mathcal{A}\}$.
3. (H_E, π_E), the subrepresentation of (H,π) generated by E.
4. H_v, the space H_E when $E = \{v\}$ for some vector $v \in H$
5. $\pi_v := \pi_E$ when $E = \{v\}$.
6. (H_v, π_v), the subrepresentation of (H,π) generated by v.
7. $H_E^{\perp}$, the orthogonal complement of H_E
8. P_E, the projection over H_E.
9. $P_{E^{\perp}}$, the projection over $H_E^{\perp}$.

Remark 3.3.2. For a tuple $\bar{v} = (v_1, \dots, v_n)$, by $P_E\bar{v}$ we denote the tuple $(P_E v_1, \dots, P_E v_n)$.

Fact 3.3.3. Let $v \in H_d$. Then v is algebraic over $\emptyset$.

Proof. If $v \in H_d$ by Theorem .3.20, there exist a sequence v_i of vectors in H_d such that $v_i \in H_i^{k_i}$, and $v = \sum_{i\geq 1} v_k$. Given that $\|v_k\| \to 0$ when $k \to \infty$, the orbit of v under any automorphism U of (H,π) is a Hilbert cube which is compact, which implies that v is algebraic. □

Fact 3.3.4 (Proposition 2.7 in [17]). Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, $A \subseteq M$ and $B \subseteq N$. If $f : A \to B$ is an elementary map, then there is an elementary map $g : acl_{\mathcal{M}}(A) \to acl_{\mathcal{N}}(B)$ extending f. Moreover, if f is onto, then so is g.

Theorem 3.3.5. *Let (H_1, π_1) and (H_2, π_2) be two representations of $\mathcal{A}$. Then $(H_1, \pi_1) \equiv (H_2, \pi_2)$ if and only if*

$$\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$$

and

$$((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$$

Proof. By Theorem 3.2.11, $(H_1, \pi_1) \equiv (H_2, \pi_2)$ if and only if $((H_1)_d, (\pi_1)_d) \equiv ((H_2)_d, (\pi_2)_d)$ and $((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$. By Lemma 3.2.17, this is equivalent to $\big((H_1)_d, (\pi_1)_d\big) \simeq \big((H_2)_d, (\pi_2)_d\big)$ and $((H_1)_e, (\pi_1)_e) \equiv ((H_2)_e, (\pi_2)_e)$. □

Theorem 3.3.6. *Let $\kappa \geq |S_{\mathcal{A}}|$ be such that $cf(\kappa) = \kappa$. Then the structure*

$$(\tilde{H}_\kappa, \tilde{\pi}_\kappa) = (H_d, \pi_d) \oplus \bigoplus_{\kappa} (H_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}})$$

is κ universal, κ homogeneous and is a monster model for $Th(H, \pi)$.

Proof. Let us denote $(\tilde{H}_\kappa, \tilde{\pi}_\kappa)$ just by $(\tilde{H}, \tilde{\pi})$.

$(\tilde{H}, \tilde{\pi}) \models Th(H, \pi)$ For every $a \in Ball_1(\mathcal{A})$, if $rank(a) = \infty$ in H_e, then $rank(a) = \infty$ in $\tilde{H}_e$ and if $rank(a) = 0$ in H_e, then $rank(a) = 0$ in $\tilde{H}_e$. By Theorem 3.3.5 $(H_e, \pi_e) \equiv (\tilde{H}_e, \tilde{\pi}_e)$. By Theorem 3.3.5, $(H, \pi) \equiv (\tilde{H}, \tilde{\pi})$.

κ-**Universality** Let $(H', \pi') \models Th(H, \pi)$ be a model with density less than κ. By Theorem 3.3.5, $(H'_d, \pi'_d) \simeq (\tilde{H}_d, \tilde{\pi}_d) \simeq (H_d, \pi_d)$. Then without loss of generality we can asume that $\pi(\mathcal{A}) = \pi(\mathcal{A})_e$. By Theorem .3.12, there exists a set I and a family $(H_i, \pi_i, v_i)_{i \in I}$ of cyclic representations such that $(H', \pi') = \bigoplus_{i \in I}(H_i, \pi_i)$. By Theorem .3.31, $(H_{v_i}, \pi_{v_i}, v_i) \simeq (L^2(\mathcal{A}, \phi_{v_i}), M_{\phi_{v_i}}, (e)_{\sim_{\phi_{v_i}}})$. Since the density of (H', π') is less than κ, the size of I is less than κ and clearly (H', π') is isomorphic to a subrepresentation of $(\tilde{H}, \tilde{\pi})$.

κ-**Homogeneity** Let U be a partial elementary map between E, $F \subseteq \tilde{H}$ with $|E| = |F| < \kappa$.

1. We can extend U to an unitary equivalence between H_E and H_F: Let a_1, $a_2 \in \mathcal{A}$ and e_1, $e_2 \in E$. Then we define $U(\pi(a_1)(e_1) + \pi(a_2)(e_2)) := \pi(a_1)(U(e_1)) + \pi(a_2)(U(e_2))$. After this, we extend this constuction continuously to H_E.
2. We can extend U to an unitary equivalence between $(H_d \oplus H_{E_e})$ and $(H_d \oplus H_{F_e})$: By Lemma 3.3.3, $(H_d \oplus H_{E_e}) \subseteq acl_{\tilde{H}}(E)$ and $(H_d \oplus H_{F_e}) \subseteq acl_{\tilde{H}}(F)$. By Fact 3.3.4, we can extend U in the desired way.

3. We can find an unitary equivalence between $(H_d \oplus H_{E_e})^\perp$ and $(H_d \oplus H_{F_e})^\perp$: Given that $|E| = |F| < \kappa$, there are two subsets C_1 and C_2 of κ such that $(H_d \oplus H_{E_e})^\perp = \bigoplus_{C_1} H_{PS_{\pi(\mathcal{A})e}}$ and $(H_d \oplus H_{E_e})^\perp = \bigoplus_{C_2} H_{PS_{\pi(\mathcal{A})e}}$. We have that $|C_1| = |C_2| = \kappa$ and therefore,
$$\bigoplus_{C_1}(H_{S_{\pi(\mathcal{A})e}}, \pi_{S_{\pi(\mathcal{A})e}}) \simeq \bigoplus_{C_2}(H_{S_{\pi(\mathcal{A})e}}, \pi_{S_{\pi(\mathcal{A})e}}).$$
Let U' an isomorphism between
$$\bigoplus_{C_1}(H_{S_{\pi(\mathcal{A})e}}, \pi_{S_{\pi(\mathcal{A})e}}) \text{ and } \bigoplus_{C_2}(H_{S_{\mathcal{A}e}}, \pi_{S_{\pi(\mathcal{A})e}}).$$

4. Let $v \in \tilde{H}_\kappa$. Then $v = v_d + v_{E_e} + v_{E_e^\perp}$, where $v_{E_e} := P_{E_e}v$, $v_d \in H_d$ and $v_{E_e^\perp} := P_{E_e^\perp}v$. Let $w := Uv_d + Uv_{E_e} + U'v_{E_e^\perp}$, and $U'' := U \oplus U'$. Then w and U'' are such that U'' is an automorphism of $\tilde{H}_\kappa$ extending U such that $U''v = w$.

□

Definition 3.3.7. Let (H_1, π_1), $(H_2, \pi_2) \models IHS_{\mathcal{A},\pi}$. Let $\epsilon > 0$. An *ϵ-perturbation* from (H_1, π_1) to (H_2, π_2), is an unitary operator $U : H_1 \to H_2$ such that for every $a \in \mathcal{A}$ we have that:

1. The operator $\pi_1(a) - U^{-1}\pi_2(a)U$ can be extended to a bounded operator on H_1 with norm less or equal to ϵ.

2. The operator $\pi_2(a) - U\pi_1(a)U^{-1}$ can be extended to a bounded operator on H_2 with norm less or equal to ϵ.

Theorem 3.3.8. *$IHS_{\mathcal{A},\pi}$ is $\aleph_0$-categorical up to the system of perturbations, i.e., for all separable (H_1, π_1), $(H_2, \pi_2) \models IHS_{\mathcal{A},\pi}$ and for all $\epsilon > 0$, there is an ϵ-perturbation U_ϵ : $(H_1, \pi_1) \to (H_2, \pi_2)$.*

Proof. Since (H_1, π_1), $(H_2, \pi_2) \models IHS_{\mathcal{A},\pi}$, (H_1, π_1) and (H_2, π_2) are elementary equivalent. By Lemma 3.2.10 (H_1, π_1) and (H_2, π_2) are approximately equivalent. So, we get the desired conclusion. □

Remark 3.3.9. Previous theorem implies that any two separable structures (H_1, π_1), $(H_2, \pi_2) \models IHS_{\mathcal{A},\pi}$ are approximately unitarily equivalent.

3.4 Types and quantifier elimination

In this section we provide a characterization of types in (H, π). The main results here are Theorem 3.4.3 and Theorem 3.4.6 that characterize types in terms of subrepresentations of (H, π) and, its consecuence, Corollary 3.4.7 that states that $IHS_{\mathcal{A},\pi}$ has quantifier elimination. As in the previous section, we denote by $(\tilde{H}, \tilde{\pi})$ the monster model for the theory $IHS_{\mathcal{A},\pi}$ as constructed in Theorem 3.3.6.

Remark 3.4.1. An automorphism U of (H,π) is a unitary operator U on H such that $U\pi(a) = \pi(a)U$ for every $a \in Ball_1(\mathcal{A})$.

Proof. Asume U is an automorphism of (H,π). It is clear that U must be a linear operator. Also, for every $v, w \in H$ and $\pi(a) \in \mathcal{A}$, we must have that $U(\pi(a)v) = \pi(a)(Uv)$ and $\langle Uv \,|\, Uw\rangle = \langle v \,|\, w\rangle$ by definition of automorphism. Therefore U must be unitary and commutes with the elements of $\pi(\mathcal{A})$. Conversely, if U is an unitary operator commuting with the elements of $\pi(\mathcal{A})$, then U is clearly an automorphism of (H,π). □

Lemma 3.4.2. *Let*

$$H_d = \bigoplus_{i\in\mathbb{Z}^+} H_i^{(k_i)}$$

be as in Theorem .3.20. Let $v \in H_i^{(k_i)}$ for some $i \in \mathbb{Z}^+$ and let $U \in Aut(H,\pi)$. Then $Uv \in H_i^{(k_i)}$.

Proof. By Theorem .3.20, $\pi(\mathcal{A})_d = \pi(\mathcal{A}) \cap \mathcal{K}(H)$ can be seen as:

$$\pi(\mathcal{A})_d = \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i^{(k_i)}).$$

By Remark 3.4.1, any automorphism $U \in \mathrm{Aut}((H,\pi))$ commutes with every element of $\pi(\mathcal{A})$, in particular with any element K of $\mathcal{K}(H_i^{(k_i)})$. Thus, if $v \in H_i^{(k_i)}$ and $K \in \mathcal{K}(H_i^{(k_i)})$, $KUv = UKv$. This implies that $Uv \in H_i^{(k_i)}$. □

Theorem 3.4.3. *Let v, $w \in \tilde{H}$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if (H_v,π_v,v) is isometrically isomorphic to (H_w,π_w,w) (see Definition .3.9 and Notation .3.11).*

Proof. Let us suppose that $tp(v/\emptyset) = tp(w/\emptyset)$. Then there is an automorphism U of $(\tilde{H},\tilde{\pi})$ such that $Uv = w$. Therefore the representations (H_v,π_v,v) and (H_w,π_w,w) are unitarily equivalent and therefore (H_v,π_v,v) is isometrically isomorphic to (H_w,π_w,w).
Conversely, let (H_v,π_v,v) be isometrically isomorphic to (H_w,π_w,w). By Theorem 3.3.6, (H_v,π_v) and (H_w,π_w) can be seen as subrepresentations of $(\tilde{H},\tilde{\pi})$. Given that (H_v,π_v,v) and (H_w,π_w,w) are isometrically isomorphic, by Theorem .3.20 and Theorem 3.3.6, the decompositions of (H_v,π_v) and (H_w,π_w) into cyclic representations are isometrically isomorphic too, and therefore $\tilde{H}_v^\perp$ and $\tilde{H}_w^\perp$ are isometrically isomorphic. Then we get an automorphism of $(\tilde{H},\tilde{\pi})$ that sends v to w, and v and w have the same type over the empty set. □

Theorem 3.4.4. *Let $v, w \in H$. Then $tp(v/\emptyset) = tp(w/\emptyset)$ if and only if $\phi_v = \phi_w$, where ϕ_v denotes the positive linear functional on $\mathcal{A}$ defined by v as in Lemma .3.25.*

Proof. Let v and $w \in H$ be such that $\mathrm{tp}(v/\emptyset)$ =$\mathrm{tp}(w/\emptyset)$. Then $\mathrm{qftp}(v/\emptyset)$ =$\mathrm{qftp}(w/\emptyset)$ and therefore, for every $a \in \mathcal{A}$, $\langle\pi(a)v|v\rangle = \langle\pi(a)w|w\rangle$. But this means that $\phi_v = \phi_w$.
Conversely, if $\phi_v = \phi_w$, by Theorem .3.14, (H_v,π_v,v) is isometrically isomorphic to (H_w,π_w,w) and by Theorem 3.4.3 $\mathrm{tp}(v/\emptyset)$ =$\mathrm{tp}(w/\emptyset)$. □

Lemma 3.4.5. *Let $E \subseteq H$, $U \in Aut(H,\pi)$. Then $U \in Aut((H,\pi)/E)$ if and only if $U \restriction (H_E, \pi_E) = Id_{(H_E,\pi_E)}$.*

Proof. Suppose that $U \restriction (H_E, \pi_E) = Id_{(H_E,\pi_E)}$. Then, U fixes H_E pointwise, and, therefore, fixes E pointwise. Conversely, suppose $U \in \mathrm{Aut}((H,\pi)/E)$. By Remark 3.4.1, U is an unitary operator that commutes with every $S \in \pi(\mathcal{A})$. Then for every $S \in \pi(\mathcal{A})$ and $v \in E$, we have that $U(Sv) = S(Uv) = Sv$. So U acts on H_E like the identity and the conclusion follows. □

Theorem 3.4.6. *Let v and $w \in \tilde{H}$ and $E \subseteq \tilde{H}$. Then $tp(v/E) = tp(w/E)$ if and only if $P_E(v) = P_E(w)$ and $tp(P_E^{\perp}(v)/\emptyset) = tp(P_E^{\perp}(w)/\emptyset)$.*

Proof. ⇒ Suppose $\mathrm{tp}(v/E) = \mathrm{tp}(w/E)$. Given that $\mathrm{tp}(v/E) = \mathrm{tp}(w/E)$, there exists $U \in \mathrm{Aut}((\tilde{H},\tilde{\pi})/E)$ such that $Uv = w$. By Lemma 3.4.5, $U \restriction (H_E, \pi_E) = Id_{(H_E,\pi_E)}$ and $U(P_E(v)) = P_E(Uv) = P_E(w)$. On the other hand, $U(P_E^{\perp}(v)) = P_E^{\perp}(w)$ and therefore $\mathrm{tp}(P_E^{\perp}(v)/\emptyset) = \mathrm{tp}(P_E^{\perp}(w)/\emptyset)$.

⇐ Asume $P_E(v) = P_E(w)$ and $\mathrm{tp}(P_E^{\perp}(v)/\emptyset) = \mathrm{tp}(P_E^{\perp}(w)/\emptyset)$. Then there exists an automoprhism U of $(\tilde{H},\tilde{\pi})$ such that $U(P_E^{\perp}(v)) = P_E^{\perp}(w)$. Let $\tilde{U} = Id_{H_E} \oplus (U \restriction \tilde{H}_E^{\perp})$. Then, by Lemma 3.4.5, $\tilde{U}$ is an automorphism of $(\tilde{H},\tilde{\pi})$ that fixes E pointwise and $Uv = w$. This implies that $\mathrm{tp}(v/E) = \mathrm{tp}(w/E)$.

□

Corollary 3.4.7. *The structure (H,π) has quantifier elimination.*

Proof. This follows from Theorem 3.4.6 that shows that types are determined by quantifier-free conditions contained in it. □

Remark 3.4.8. Note that the quantifier elimination result that we proved is uniform. That is, types are isolated by the conditions $\langle \dot{a}x \mid x \rangle$ for $a \in \mathcal{A}$ no matter the particular non-degenerate representation (H,π) we choose.

Recall that the weak* topology in $\mathcal{A}'$ (the Banach dual Algebra of $\mathcal{A}$) is the coarsest topology in $\mathcal{A}'$ such that for every $a \in A$, the function $F_a : \mathcal{A}' \to \mathbb{C}$ is continuous, where $F_a(\phi) = \phi(a)$ for $a \in \mathcal{A}$ and $\phi \in \mathcal{A}'$.

3.5 A model companion for $IHS_{\mathcal{A}}$

In this section we prove that the theory $IHS_{\mathcal{A}}$ has a model companion (Theorem 3.5.3). This result goes in the same direction as results from Ben Yaccov and Usvyatsov ([37]) and Berenstein and Henson ([8]).

Definition 3.5.1. Let $EIHS_{\mathcal{A}}$ be the theory of a representation (H,π) such that no element of $\mathcal{A}$ acts as a compact operator.

Theorem 3.5.2. *For every Hilbert space representation $(H,\pi) \models IHS_{\mathcal{A}}$, there is a Hilbert space representation $(H',\pi') \models EIHS_{\mathcal{A}}$, such that $(H,\pi) \subseteq (H',\pi')$.*

Proof. Let $H = H_d \oplus H_e$ as in Definition 3.2.16. We define:

$$H' := (\bigoplus_{\omega} H_d) \oplus H_e$$

and for every $a \in \mathcal{A}$ such that $\pi(a)$ is compact in H,

$$\pi'(a) : (\bigoplus_{\omega} \pi(a)) \oplus 0$$

and

$$\pi'(a) := 0 \oplus \pi(a)$$

for every $a \in \mathcal{A}$ such that $\pi(a)$ is non-compact in H.
Then, (H',π') is clearly a representation of $\mathcal{A}$. Let $a \in \mathcal{A}$ such that $\pi(a)$ is compact in H. Since the non-zero eigenvalues of $\pi'(a)$ have infinite dimensional eigenspaces, the operator $\pi'(a)$ cannot be compact in (H',π'). Therefore all the elements of $\mathcal{A}$ that acted compactly on H no longer act compactly on H'. Since the elements of $\mathcal{A}$ that acted non-compactly on H still act non-compactly on H', no element of $\mathcal{A}$ act compactly on H' and therefore, $(H',\pi') \models EIHS_{\mathcal{A}}$. □

Corollary 3.5.3. *$EIHS_{\mathcal{A}}$ is a model companion for $IHS_{\mathcal{A}}$.*

Proof. Clearly, every model of $EIHS_{\mathcal{A}}$ is a model of $IHS_{\mathcal{A}}$. On the other hand, by Theorem 3.5.2, every model of $IHS_{\mathcal{A}}$ can be (non elementarily) embedded in a model of $EIHS_{\mathcal{A}}$. These previous fact show that $EIHS_{\mathcal{A}}$ is a companion for $IHS_{\mathcal{A}}$. Since by Corollary 3.4.7 the theory $EIHS_{\mathcal{A}}$ is model complete, the theory $EIHS_{\mathcal{A}}$ is a model companion for $IHS_{\mathcal{A}}$. □

Definition 3.5.4. Given a C^*-algebra $\mathcal{A}$, a model of $EIHS_{\mathcal{A}}$ is called a *generic representation of $\mathcal{A}$.*

Theorem 3.5.5. *Let (H,π) be a generic representation of $\mathcal{A}$. The stone space $S_1(Th(H,\pi))$ (i.e. the set of types of vectors of norm less than or equal to 1) with the logic topology is homeomorphic to the quasi state space $Q_{\mathcal{A}}$ with the weak* topology.*

Proof. We consider types of vectors with norm less than or equal to 1. Similarly, we consider positive linear functionals with norm less than or equal to 1, that is, the quasi state space $Q_{\mathcal{A}}$. By Theorem 3.4.4, types of vectors in H are determined by the corresponding positive linear functionals, explicitly $v \to \phi_v(a) := \langle \pi(a)v \mid v \rangle$. Conversely, given a quasi state ϕ, by Gelfand-Naimark-Segal Construction (see Theorem .3.30), there exists a representation (H,π) of $\mathcal{A}$ and a vector $v \in H$ such that $\phi = \phi_v$. Without loss of generaltiy, we can assume

that (H,π) is generic. So, there is a bijection between $S_1(Th(H,\pi))$ and $Q_{\mathcal{A}}$. To prove bicontinuity, let $h : S_1(Th(H,\pi)) \to Q_{\mathcal{A}}$ be the previously defined bijection. Let X be a weak* basic open set in $Q_{\mathcal{A}}$; then there exists an open sets $V_1,\dots,V_k \subseteq \mathbb{C}$ and elements $a_1,\dots,a_k \in \mathcal{A}$ such that for every $\phi \in Q_{\mathcal{A}}$, we have that $\phi \in X$ if and only if $\phi(a_i) \in V_i$. For $\phi \in X$ let v_ϕ be a cyclic vector such that $\phi = \phi_{v_\phi}$. Then for every $\phi \in X$ and $i = 1,\dots,k$, $\langle \pi(a_i)v_\phi \mid v_\phi\rangle \in V_i$ but this condition defines an open set in $S_1(Th(H,\pi))$.
Conversely, by quantifier elimination, every basic open sets X in the logic topology in $S_1(Th(H,\pi))$ can be expresed as finite intersection of sets with the form:

$$\{p \in S_1(Th(H,\pi)) \mid v_\phi \models p \Rightarrow \langle \pi(a)v_\phi \mid v_\phi\rangle \in V\}$$

where $V \subseteq \mathbb{C}$ open. Each of this sets is in correspondence by h with a set of the form

$$\{\phi \in Q_{\mathcal{A}} \mid \langle \pi(a)v_\phi \mid v_\phi\rangle \in V\}$$

which defines an open set in $Q_{\mathcal{A}}$. □

3.6 Definable and algebraic closures

In this section we give a characterization of definable and algebraic closures. Gelfand-Naimark-Segal Construction is a tool for understanding definable closures (see Theorem .3.30).

Theorem 3.6.1. *Let $E \subseteq H$. Then $dcl(E) = H_E$*

Proof. From Fact 3.4.5, it is clear that $H_E \subseteq dcl(E)$. On the other hand, if $v \notin H_E$, let $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$ and $|\lambda| = 1$. Then, the operator $U := Id_{H_E} \oplus \lambda Id_{H_E^\perp}$ is an automorphism of (H,π) fixing E such that $Uv \neq v$. □

Definition 3.6.2. Given a representation $\pi : \mathcal{A} \to B(H)$, we define:

- The *essential part* of π is the C^*-algebra homomorphism,

  $\pi_e := \rho \circ \pi : \mathcal{A} \to B(H)/\mathcal{K}(H)$

  of $\pi(\mathcal{A})$, where ρ is the canonical proyection of $B(H)$ onto the Calkin Algebra $B(H)/\mathcal{K}(H)$.

- The *discrete part* of π is the restriction,

  $$\begin{aligned}\pi_d : ker(\pi_e) &\to \mathcal{K}(H)\\ a &\to \pi(a)\end{aligned}$$

- The *discrete part* of $\pi(\mathcal{A})$ is defined in the following way:

  $\pi(\mathcal{A})_d := \pi(\mathcal{A}) \cap \mathcal{K}(H)$.

- The *essential part* of $\pi(\mathcal{A})$ is the image $\pi(\mathcal{A})_e$ of $\pi(\mathcal{A})$ in the Calkin Algebra.
- The *essential part* of H is defined in the following way:

  $H_e := \ker(\pi(\mathcal{A})_d)$

- The *discrete part* of H is defined in the following way:

  $H_d := \ker(\pi(\mathcal{A})_d)^\perp$

Corollary 3.6.3. *For any index set I, $(H,\pi) \equiv (H,\pi) \oplus \bigoplus_{i\in I}(H_e,\pi_e)$.*

Proof. By Fact 3.2.11. □

Lemma 3.6.4. *Let $v \in H_e$. Then v is not algebraic over $\emptyset$.*

Proof. Let $\kappa > 2^{\aleph_0}$ and consider $(H,\pi) \oplus \bigoplus_{i\in I}(H_e,\pi_e)$. By Corollary 3.6.3 this structure is elementary equivalent to (H,π). Then, there are κ vectors v_i for $i < \kappa$ such that every v_i has the same type over $\emptyset$ as v. This means that the orbit of v under the automorphisms of (H,π) is unbounded and therefore v is not algebraic over the emptyset. □

Lemma 3.6.5. *Let $v \in H$ such that $v_e \neq 0$. Then v is not algebraic over $\emptyset$.*

Proof. Clear from previous Lemma 3.6.4. □

Now, we describe the algebraic closure of $\emptyset$:

Theorem 3.6.6. $acl(\emptyset) = H_d$

Proof. By Lemma 3.3.3, $H_d \subseteq acl(\emptyset)$ and, by Lemma 3.6.5, $acl(\emptyset) \subseteq H_d$. □

Theorem 3.6.7. *Let $E \subseteq H$. Then $acl(E)$ is the Hilbert subspace of H generated by $dcl(E)$ and $acl(\emptyset)$.*

Proof. Let G be the Hilbert subspace of H generated by $dcl(E)$ and $acl(\emptyset)$. It is clear that $G \subseteq acl(E)$. Let $v \in acl(E)$. By Lemma 3.3.3, $v_d \in acl(\emptyset)$, and by Theorem 3.6.1 and Lemma 3.6.4, $v_e \in dcl(E) \setminus acl(\emptyset)$. Then $v_e \in dcl(E)$ and $acl(E) \subseteq G$. □

3.7 Forking and stability

In this section we give an explicit characterization of non-forking and prove that $Th(H,\pi)$ is stable. Henson and Iovino in [25], observed that a Hilbert space expanded with a family of bounded operators is stable. Here, we give an explicit description of non-forking and show that the theory is superstable.

Definition 3.7.1. Let E, F, $G \subseteq H$. We say that E is *independent* from G over F if for all $v \in E$ $P_{acl(F)}(v) = P_{acl(F\cup G)}(v)$ and denote it by $E \mathop{\smile\!\!\!\!\!\mid}^*_F G$.

Remark 3.7.2. Let $\bar{v}$, $\bar{w} \in H^n$ and $E \subseteq H$. Then it is easy to see that:

- $\bar{v}$ is independent from $\bar{w}$ over $\emptyset$ if and only if for every $j,k = 1,\dots,n$, $H_{(v_j)_e} \perp H_{(w_k)_e}$.
- $\bar{v}$ is independent from $\bar{w}$ over E if and only if for every $j,k = 1,\dots,n$, $H_{P_E^\perp(v_j)_e} \perp H_{P_E^\perp(w_k)_e}$.
- $\bar{v} \in H^n$ and E, $F \subseteq H$. Then $\bar{v} \mathop{\smile\!\!\!\!\!\mid}^*_E F$ if and only if for every $j = 1,\dots,n$ $v_j \mathop{\smile\!\!\!\!\!\mid}^*_E F$ that is, for all $j = 1,\dots,n$ $P_{acl(E)}(v_j) = P_{acl(E\cup F)}(v_j)$.

Theorem 3.7.3. *Let $E \subseteq F \subseteq H$, $p \in S_n(E)$, $q \in S_n(F)$ and $\bar{v} = (v_1,\dots,v_n)$, $\bar{w} = (w_1,\dots,w_n) \in H^n$ be such that $p = tp(\bar{v}/E)$ and $q = tp(\bar{w}/F)$. Then q is an extension of p such that $\bar{w} \mathop{\smile\!\!\!\!\!\mid}^*_E F$ if and only if the following conditions hold:*

1. *For every $j = 1,\dots,n$, $P_{acl(E)}(v_j) = P_{acl(F)}(w_j)$*
2. *For every $j = 1,\dots,n$, $(H_{P^\perp_{acl(E)}v_j}, \pi_{P^\perp_{acl(E)}v_j}, P^\perp_{acl(E)}v_j)$ is isometrically isomorphic to $(H_{P^\perp_{acl(F)}w_j}, \pi_{P^\perp_{acl(F)}w_j}, P^\perp_{acl(F)}w_j)$*

Proof. Clear from Theorem 3.4.3 and Remark 3.7.2 □

Remark 3.7.4. Recall that for every $E \subseteq H$ and $v \in H$, $P^\perp_{acl(E)}v = (P_E^\perp v)_e$.

Theorem 3.7.5. $\mathop{\smile\!\!\!\!\!\mid}^*$ *is a freeness relation.*

Proof. By Remark 3.7.2, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character Let $v \in H$ and $E \subseteq H$. Let $w = (P_{acl(E)}(v))_e$. Then there exist a sequence of $(l_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, a sequence of finite tuples $(a_1^k,\dots,a_{l_k}^k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e_1^k,\dots,e_{l_k}^k)_{k\in\mathbb{N}} \subseteq E$ such that if $w_k := \sum_{j=1}^{l_k} \pi(a_j^k)e_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$. Let $E_0 = \{e_j^k \mid j = 1,\dots,l_k \text{ and } k \in \mathbb{N}\}$. Then $v \mathop{\smile\!\!\!\!\!\mid}^*_{E_0} E$ and $|E_0| = \aleph_0$.

Finite character We show that for $v \in H$, $E, F \subseteq H$, $v \mathop{\smile\!\!\!\!\!\mid}^*_E F$ if and only if $v \mathop{\smile\!\!\!\!\!\mid}^*_E F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\mathop{\smile\!\!\!\!\!\mid}^*_E F$. Let $w = P_{acl(E\cup F)}(v) - P_{acl(E)}(v)$. Then $w \in \mathrm{acl}(E \cup F)\backslash\mathrm{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}^2$, a sequence of finite tuples $(a_1^k,\dots,a_{l_k+n_k}^k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e_1^k,\dots,e_{l_k}^k, f_1^k,\dots,f_{n_k}^k)_{k\in\mathbb{N}}$ such that $(e_1^k,\dots,e_{l_k}^k) \subseteq E$, $(f_1^k,\dots,f_{n_k}^k)_{k\in\mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} \pi(a_j^k)e_i^k + \sum_{j=1}^{n_k} \pi(a_{l_k+j}^k)f_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$.

Since $v \not\mathop{\smile\!\!\!\!\!\mid}^*_E F$, then $w = P_{acl(E\cup F)}(v) - P_{acl(E)}(v) \neq 0$. Let $\epsilon = \|w\| > 0$. Then, there is k_ϵ such that if $k \geq k_\epsilon$ then $\|w - w_k\| < \epsilon$. Let $F_0 := \{f_1^1,\dots,f_{k_\epsilon}^{n_{k_\epsilon}}\}$, then F_0 is a finite subset such that $v \not\mathop{\smile\!\!\!\!\!\mid}^*_E F_0$.

Transitivity of independence Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \downarrow^*_E G$ then $P_{acl(E)}(v) = P_{acl(G)}(v)$. It is clear that $P_{acl(E)}(v) = P_{acl(F)}(v) = P_{acl(G)}(v)$ so $v \downarrow^*_E F$ and $v \downarrow^*_F G$. Conversely, if $v \downarrow^*_E F$ and $v \downarrow^*_F G$, we have that $P_{acl(E)}(v) = P_{acl(F)}(v)$ and $P_{acl(F)}(v) = P_{acl(G)}(v)$. Then $P_{acl(E)}(v) = P_{acl(G)}(v)$ and $v \downarrow^*_E G$.

Symmetry It is clear from Remark 3.7.2.

Invariance Let U be an automorphism of (H, π). Let $\bar{v} = (v_1, \dots, v_n)$,$\bar{w} = (w_1, \dots, w_n) \in H^n$ and $E \subseteq H$ be such that $\bar{v} \downarrow^*_E \bar{w}$. By Remark 3.7.2, this means that for every $j,\ k = 1, \dots, n$ $H_{P^\perp_{acl(E)}(v_j)} \perp H_{P^\perp_{acl(E)}(w_k)}$. It follows that for every $j,\ k = 1, \dots, n$ $H_{P^\perp_{acl(UE)}(Uv_j)} \perp H_{P^\perp_{acl(UE)}(Uw_k)}$ and, again by Remark 3.7.2, $Uv \downarrow^*_{acl(UE)} Uw$.

Existence Let $(\tilde{H}, \tilde{\pi})$ be the monster model and let $E \subseteq F \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(E)$, there exists $q \in S_n(F)$ such that q is a $\downarrow^*$-independent extension of p.

Case $n = 1$ Let $v \in \tilde{H}$ be such that $p = tp(v/E)$ and let

$$(H', \pi', u) := (H_{(P^\perp_{acl(E)} v)}, \pi_{(P^\perp_{acl(E)} v)}, P^\perp_{acl(E)} v)$$

Then, by Fact 3.2.11, the model $(\hat{H}, \hat{\pi}) := (H, \pi) \oplus (H', \pi')$ is an elementary extension of (H, π). Let $v' := P_{acl(E)} v + P^\perp_{acl(E)} v_d + u \in \hat{H}$. Then, by Theorem 3.7.3, the type $tp(v'/F)$ is a $\downarrow^*$-independent extension of $tp(v/E)$.

Induction step Now, let $\bar{v} = (v_1, \dots, v_n, v_{n+1}) \in \tilde{H}^{n+1}$. By induction hypothesis, there are $v'_1, \dots, v'_n \in H$ such that $tp(v'_1, \dots, v'_n/F)$ is a $\downarrow^*$-independent extension of $tp(v_1, \dots, v_n/E)$. Let U be an automorphism of the monster model fixing E pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v'_j$. Let $v'_{n+1} \in \tilde{H}$ be such that $tp(v'_{n+1}/F v'_1 \cdots v'_n)$ is a $\downarrow^*$-independent extension of $tp(U(v_{n+1})/E v'_1, \cdots v'_n)$. Then, by transitivity, $tp(v'_1, \dots, v'_n, v'_{n+1}/F)$ is a $\downarrow^*$-independent extension of $tp(v_1, \dots, v_n, v_{n+1}/E)$.

Stationarity Let $(\tilde{H}, \tilde{\pi})$ be the monster model and let $E \subseteq F \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(E)$, if $q \in S_n(F)$ is a $\downarrow^*$-independent extension of p to F then $q = p'$, where p' is the $\downarrow^*$-independent extension of p to F built in the proof of existence.

Case $n = 1$ Let $v \in H$ be such that $p = tp(v/E)$, and let $q \in S(F)$ and $w \in H$ be such that $w \models q$. Let v' be as in previous item. Then, by Theorem 3.7.3 we have that:

1. $P_{acl(E)} v = P_{acl(F)} v' = P_{acl(F)} w$
2. $(H_{P^\perp_{acl(E)} v}, \pi_{P^\perp_{acl(E)} v}, P^\perp_{acl(E)} v)$ is isometrically isomorphic to both

$$(H_{P^\perp_{acl(F)} w}, \pi_{P^\perp_{acl(F)} w}, P^\perp_{acl(F)} w)$$

and

$$(H_{P^{\perp}_{acl(F)}v'}, \pi_{P^{\perp}_{acl(F)}v'}, P^{\perp}_{acl(F)}v')$$

This means that $P_{acl(F)}v' = P_{acl(F)}w$ and $(H_{P^{\perp}_{acl(F)}w}, \pi_{P^{\perp}_{acl(F)}w}, P^{\perp}_{acl(F)}w)$ is isometrically isomorphic to $(H_{P^{\perp}_{acl(F)}v'}, \pi_{P^{\perp}_{acl(F)}v'}, P^{\perp}_{acl(F)}v')$ and, therefore $q = tp(v'/F) = p'$.

Induction step Let $\bar{v} = (v_1, \ldots, v_n, v_{n+1})$, $\bar{v}' = (v_1', \ldots, v_n, v_{n+1}')$ and $\bar{w} = (w_1, \ldots, w_{n+1}) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $tp(v_1', \ldots, v_n'/F)$ and $tp(w_1, \ldots, w_n/F)$ are $\perp\!\!\!\!\perp^*$-independent extensions of $tp(v_1, \ldots, v_n/E)$. By induction hypothesis, $tp(v_1', \ldots, v_n'/F) = tp(w_1, \ldots, w_n/F)$. Let U be an automorphism of the monster model fixing E pointwise such that for every $j = 1, \ldots, n$, $U(v_j) = v_j'$ and let U' an automorphism of the monster model fixing F pointwise such that for every $j = 1, \ldots, n$, $U'(v_j') = w_j'$. Again by transitivity, $tp(U^{-1}(v_{n+1}')/Fv_1 \cdots v_n)$ and $tp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ are $\perp\!\!\!\!\perp^*$-independent extensions of $tp(v_{n+1}/Ev_1, \cdots v_n)$. By the case $n = 1$ $tp(U^{-1}(v_{n+1}')/Fv_1 \cdots v_n) = tp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ and therefore

$$p' = tp(v_1', \ldots, v_n'v_{n+1}'/F) = tp(w_1, \ldots, w_n, w_{n+1}/F) = q.$$

□

Fact 3.7.6 (Theorem 14.14 in [21]). A first order continuous logic theory T is stable if and only if there is an independence relation $\perp\!\!\!\!\perp^*$ satisfying local character, finite character of dependence, transitivity, symmetry, invariance, existence and stationarity. In that case the relation $\perp\!\!\!\!\perp^*$ coincides with non-forking.

Theorem 3.7.7. *The theory T_π is superstable and the relation $\perp\!\!\!\!\perp^*$ agrees with non-forking.*

Proof. By Fact 3.7.6, T_π is stable and the relation $\perp\!\!\!\!\perp^*$ agrees with non-forking. To prove superstability, we have to show that for every $\bar{v} = (v_1, \ldots, v_n) \in H$, every $F \subseteq H$ and every $\epsilon > 0$, there exist a finite $F_0 \subseteq F$ and $\bar{v}' = (v_1', \ldots, v_n') \in H^n$ such that $\|v_j - v_j'\| < \epsilon$ and $v_j' \perp\!\!\!\!\perp_{F_0} F$ for every $j \leq n$. As in the proof of local character, for $j = 1, \ldots, n$ let $(a_1^{jk}, \ldots, a_{l_k}^{jk})_{k \in \mathbb{N}}$, $(e_1^{jk}, \ldots, e_{l_k}^{jk})_{k \in \mathbb{N}}$, $w_j := (P_{acl(E)}(v_j))_e$ and $(w_j^k)_{k \in \mathbb{N}}$ be such that $w_j^k := \sum_{s=1}^{l_k} \pi(a_s^{jk})\ e_s^{jk}$ for $k \in \mathbb{N}$, and $w_j^k \to w_j$. For $j = 1, \ldots, n$, let $K_j \in \mathbb{N}$ be such that $\|w_j - w_j^{K_j}\| < \epsilon$, let $v_j' := (P_{acl(E)}v_j)_d + w_j^{K_j}$ and let $F_0^j = \{e_s^k \mid k \leq K_j \text{ and } s = 1, \ldots, l_k\}$. If we define $F_0 := U_{j=1}^n F_0^j$, then for every $j = 1, \ldots, n$ we have that $v_j' \perp\!\!\!\!\perp^*_{F_0} F$, $|F_0| < \aleph_0$ and $\|v_j - v_j'\| < \epsilon$. □

Recall that a canonical base for a type p is a minimal set over which p does not fork. In general, this smallest tuple is an imaginary, but in Hilbert spaces it corresponds to a tuple of real elements. Next theorem gives an explicit description of canonical bases for types in the structure, again we get a tuple of real elements.

Theorem 3.7.8. *Let $\bar{v} = (v_1, \ldots, v_n) \in H^n$ and $E \subseteq H$. Then*

$Cb(tp(\bar{v}/E)) := \{(P_E v_1, \ldots, P_E v_n)\}$ is a canonical base for the type $tp(\bar{v}/E)$

Proof. First of all, we consider the case of a 1-tuple. By Theorem 3.7.3 $tp(v/E)$ does not fork over $Cb(tp(v/E))$. Let $(v_k)_{k<\omega}$ a Morley sequence for $tp(v/E)$. We have to show that $P_E v \in dcl((v_k)_{k<\omega})$. By Theorem 3.7.3, for every $k < \omega$ there is a vector w_k such that $v_k = P_E v + w_k$ and $w_k \perp acl(\{P_E v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_e$ and for all j, $k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v'_k := \frac{v_1 + \cdots + v_k}{n} = P_E v + \frac{w_1 + \cdots + w_k}{n}$. Then for every $k < \omega$, $v'_k \in dcl((v_k)_{k<\omega})$. Since $v'_k \to P_E v$ when $k \to \infty$, we have that $P_E v \in dcl((v_k)_{k<\omega})$.
For the case of a general n-tuple, by Remark 3.7.2, it is enough to repeat the previous argument in every component of $\bar{v}$. □

Recall that a theory is said to be uniformly finitely based if for every $\epsilon > 0$ $\exists N \in \mathbb{N}$ such that if $(v_k)_{k<\omega}$ is a Morley sequence for $tp(v/E)$ then $P_E(v) \in dcl_\epsilon(v_1, \ldots, v_N)$, where $dcl_\epsilon(v_1, \ldots, v_N)$ is the set of vectors with distance to $dcl(v_1, \ldots, v_N)$ less than ϵ.

Corollary 3.7.9. *$IHS_{\mathcal{A},\pi}$ is uniformly finitely based.*

Proof. Clear by previous theorem. □

3.8 Orthogonality and domination

In this section, we characterize domination and orthogonality of types in terms of similar relationships between positive linear functionals on $\mathcal{A}$. These are the statements Theorem 3.8.8 and Theorem 3.8.12. For a complete description of the relation of domination see [10], Definition 5.6.4.

Theorem 3.8.1. *Let $v, w \in H$. Then (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w) if and only if $\phi_v \leq \phi_w$ (see Definition .3.33).*

Proof. Suppose (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w). Then there exists a vector $v' \in H_w$ such that $(H_v, \pi_v, v) \simeq (H_{v'}, \pi_{v'}, v')$. By Radon Nikodim Theorem for rings of operators (see [16]), there exists a bounded positive operator P : $(H_w, \pi_w, w) \to (H_{v'}, \pi_{v'}, v')$ such that $Pw = v'$ and P commutes with every element of $\pi_v(\mathcal{A})$. Let $\gamma = \|P\|^2$. Then, for every positive element $a \in \mathcal{A}$, $\phi_v(a) = \phi_{v'}(a) = \langle \pi(a)v' \mid v' \rangle = \langle \pi(a)Pw \mid Pw \rangle = \langle P^*\pi(a)Pw \mid w \rangle = \langle \pi(a)\|P\|^2 w \mid w \rangle \leq \gamma \langle \pi(a)w \mid w \rangle = \gamma\phi_w(a)$ which means that $\gamma\phi_w - \phi_v$ is positive and $\phi_v \leq \phi_w$.
The converse is Corollary 3.3.8 in [31]. □

Lemma 3.8.2. *Let $v, w \in H$. If $\phi_v \perp \phi_w$ (see Definition .3.33), then (H_v, π_v, v) is not isometrically isomorphic to any subrepresentation of (H_w, π_w, w).*

Proof. Suppose $\phi_v \perp \phi_w$, and (H_v, π_v, v) is isometrically isomorphic to subrepresentation of (H_w, π_w, w). By Theorem 3.8.1 $\phi_v \leq \phi_w$; let $\gamma > 0$ be a real number such that $\gamma\phi_w - \phi_v$ is a bounded positive functional and let $u \in H$ be such that $\phi_u = \gamma\phi_w - \phi_v$, which is possible by GNS Theorem. Then $\phi_v = \gamma\phi_w - \phi_u$, and $\|\phi_w - \phi_v\| = \|\phi_w - \gamma\phi_w + \phi_u\| = \|(1-\gamma)\phi_w + \phi_u\| = |1-\gamma|\|\phi_w\| + \|\phi_u\| \neq \|\phi_w\| + \|\phi_v\|$, but this contradicts $\phi_v \perp \phi_w$. □

Here, a few facts that will be needed to prove Theorem 3.8.4:

Remark 3.8.3. Recall that two representations are said to be *disjoint* if they do not have any common subrepresentation up to isometric isomorphism.

Theorem 3.8.4. *Let $v, w \in H$. $\phi_v \perp \phi_w$ if and only if no subrepresentation of (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w).*

Proof. Suppose $\phi_v \perp \phi_w$. By Lemma .3.35, if $(H_{v'}, \pi_{v'}, v')$ is a subrepresentation of (H_v, π_v, v) and $(H_{w'}, \pi_{w'}, w')$ is a subrepresentation of (H_w, π_w, w), then $\phi_{v'} \perp \phi_{w'}$, by Lemma 3.8.2, $(H_{v'}, \pi_{v'}, v')$ is not isometrically isomorphic to $(H_{w'}, \pi_{w'}, w')$, and the conclusion follows.
Conversely, suppose no subrepresentation of (H_v, π_v, v) is isometrically isomorphic to a subrepresentation of (H_w, π_w, w). Then the representations (H_v, π_v) and (H_w, π_w) are disjoint. By Fact .3.8, there is a projection $P \in \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that $PP_v = P_v$ and $(I-P)P_w = P_w$. Then, $\phi_v(I-P) = \langle (I-P)v \mid v\rangle = \langle (v - PP_v v) \mid v\rangle = \langle (v-v) \mid v\rangle = 0$. On the other hand, $\phi_w(P) = \langle Pw \mid w\rangle = \langle w - (w - Pw) \mid w\rangle = \langle w - (I-P)w \mid w\rangle = \langle w - (I-P)P_w w \mid w\rangle = \langle w - P_w w \mid w\rangle = \langle w - w \mid w\rangle = 0$. By Fact .3.5 and Theorem .2.17, the projection P is strongly approximable by positive elements in $\pi(\mathcal{A})$ and therefore, for $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi_v(e-a) < \epsilon$ and $\phi_w(a) < \epsilon$. By Fact .3.34, $\phi_v \perp \phi_w$. □

Definition 3.8.5. Let $A \subseteq H$ and $p, q \in S_n(A)$. We say that p is *almost orthogonal* to q ($p \perp^a q$) if for all $\bar{a} \models p$ and $\bar{b} \models q$ $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$.

Definition 3.8.6. Let $A \subseteq \mathfrak{B}$ and $p \in S_n(A)$ and $q \in S_n(B)$ two stationary types. We say that p is *orthogonal* to q ($p \perp q$) if for all $B \supseteq A \cup B$, $p_B \supseteq p$ non-forking extension, and $q_B \supseteq q$ non-forking extension, $p_B \perp^w q_B$

Lemma 3.8.7. *Let $p, q \in S_1(\emptyset)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\phi_{v_e} \perp \phi_{w_e}$.*

Proof. Suppose $p \perp^a q$. By Remark 3.7.2, this implies that $H_{v_e} \perp H_{w_e}$ for all $v \models p$ and $w \models q$. Let $v \models p$ and $w \models q$. Then no subrepresentation of $(H_{v_e}, \pi_{v_e}, v_e)$ is isometrically isomorphic to any subrepresentation of $(H_{w_e}, \pi_{w_e}, w_e)$. By Lemma 3.8.4, this implies that $\phi_{v_e} \perp \phi_{w_e}$.
Conversely, if $p \not\perp^a q$ there are v, $w \in H$ such that $v \models p$, $w \models q$ and $H_{v_e} \not\perp H_{w_e}$. This implies that there exist elements a_1, $a_2 \in \mathcal{A}$ such that $\pi(a_1)v_e \not\perp \pi(a_2)w_e$. This means that $0 \neq \langle \pi(a_1)v_e \mid \pi(a_2)w_e\rangle = \langle v_e \mid \pi(a_1^* a_2)w_e\rangle$. So, we can assume that there exists an element

$a \in \mathcal{A}$ such that $v_e \not\perp \pi(a)w_e$. Since $v_e = P_{w_e}v_e + P^{\perp}_{w_e}v_e$ and $P_{w_e}v_e \neq 0$, we can prove that $\phi_{P_{w_e}v_e} \leq \phi_{v_e}$ by using a procedure similar to the one used in the proof of Theorem 3.8.1 and, since $P_{w_e}v_e \in H_{w_e}$, we get $\phi_{P_{w_e}v_e} \leq \phi_{w_e}$. By Lemma .3.35, this implies that $\phi_{v_e} \not\perp \phi_{w_e}$. □

Theorem 3.8.8. *Let $E \subseteq H$. Let $p, q \in S_1(E)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp^a_E q$ if and only if $\phi_{P^{\perp}_E(v_e)} \perp \phi_{P^{\perp}_E(w_e)}$*

Proof. Clear by Lemma 3.8.7. □

Theorem 3.8.9. *Let $E \subseteq H$. Let $p, q \in S_1(E)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Assume $p \perp^a q$, $E \subseteq F \subseteq H$ are small subsets of the monster model and p', $q' \in S_1(F)$ are non-forking extensions of p and q respectively. Let v, $w \in H$ be such that $v \models p'$ and $w \models q'$, then $\phi_{P^{\perp}_F(v_e)} = \phi_{P^{\perp}_E v_e} \perp \phi_{P^{\perp}_E w_e} = \phi_{P^{\perp}_F(w_e)}$. By Lemma 3.8.7, this implies that $p' \perp^a q'$. Therefore $p \perp q$.
The converse is trivial. □

Definition 3.8.10. Let A, $\mathfrak{B}$ be small subsets of $\tilde{H}$ and $p \in S_n(A)$ and $q \in S_n(B)$ two stationary types. We say that p *dominates* q over a set $C \supseteq A \cup B$ ($p \rhd_C q$) if there exist v $w \in \tilde{\mathcal{H}}$ such that $tp(v/C)$ is a non-forking extension of p, $tp(w/C)$ is a non-forking extension of q and for all $D \supseteq C$ if $v \mathop{\smile\!\!\!\!\!\;|}^*_C D$ then $w \mathop{\smile\!\!\!\!\!\;|}^*_C D$

Lemma 3.8.11. *Let $p, q \in S_1(\emptyset)$ and let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \rhd_\emptyset q$ if and only if $\phi_{w_e} \leq \phi_{v_e}$.*

Proof. Suppose $p \rhd_\emptyset q$. Suppose that v' and w' are such that $v' \models p$, $w' \models q$ and if $v' \mathop{\smile\!\!\!\!\!\;|}^*_\emptyset E$ then $w' \mathop{\smile\!\!\!\!\!\;|}^*_\emptyset E$ for every E. Then for every $E \subseteq H$

$$P_E v'_e = 0 \Rightarrow P_E w'_e = 0$$

This implies that $w'_e \in H_{v'_e}$, and $H_{w'_e} \subseteq H_{v'_e}$. By Theorem 3.8.1, $\phi_{w_e} = \phi_{w'_e} \leq \phi_{v'_e} = \phi_{v_e}$.
For the converse, suppose $\phi_{w_e} \leq \phi_{v_e}$. Then, by Theorem 3.8.1 H_{w_e} is isometrically isomorphic to a subrepresentation of H_{v_e}, which implies that there is $w' \in H_v$ such that $w' \models tp(w/\emptyset)$ and for every $E \subseteq H$

$$P_E v_e = 0 \Rightarrow P_E w'_e = 0$$

This means than $tp(w/\emptyset) \lhd_\emptyset tp(v/\emptyset)$. □

Theorem 3.8.12. *Let E, F and G be small subsets of $\tilde{H}$ such that E, $F \subseteq G$ and $p \in S_1(E)$ and $q \in S_1(F)$ be two stationary types. Then $p \rhd_G q$ if and only if there exist v, $w \in \tilde{H}$ such that $tp(v/G)$ is a non-forking extension of p, $tp(w/G)$ is a non-forking extension of q and $\phi_{P^{\perp}_{acl(G)} w_e} \leq \phi_{P^{\perp}_{acl(G)} v_e}$.*

Proof. Clear by Lemma 3.8.11. □

3.9 Examples

3.9.1 The structure $(H, \mathcal{K}(H))$

Let H be a separable infinite dimensional Hilbert space and let $\mathcal{K}(H)$ be the set of compact operators on H. $\mathcal{K}(H)$ is a C^*-algebra and $(H, \mathcal{K}(H))$ a representation of $\mathcal{K}(H)$ in which the C^*-algebra homomorphism is the identity on $\mathcal{K}(H)$ (denoted by $Id_{\mathcal{K}(H)}$. In this case, every $a \in \mathcal{K}(H)$ acts as a compact operator and therefore $H = H_d$.

Theorem 3.9.1. *If H is infinite dimensional, there are countably many models of $Th(H, \mathcal{K}(H))$ with density $\aleph_0$.*

Proof. Let G be another separable Hilbert space. For $a \in \mathcal{A}$ and $v \in G$, let $\pi_G(a)v := 0$. Let $H' = H \oplus G$ and let $\pi' := Id_{\mathcal{K}(H)} \oplus \pi_G$. Then, $(H', \pi') \equiv (H, \mathcal{K}(H))$. Since there are countably many separable Hilbert spaces, $Th(H, \mathcal{K}(H))$ has countably many models with density $\aleph_0$. Note that all of these representations are degenerate and this coincides with the fact that $\mathcal{K}(H)$ is not unital. □

Theorem 3.9.2. *If H is infinite dimensional, then $Th(H, \mathcal{K}(H))$ is λ-categorical for $\lambda > \aleph_0$.*

Proof. As in the proof of previous theorem, let G is any Hilbert space of density character λ. For $a \in \mathcal{A}$ and $v \in G$, let $\pi_G(a)v := 0$. Let $H' = H \oplus G$ and let $\pi' := Id_{\mathcal{K}(H)} \oplus \pi_G$. Then, $(H', \pi') \equiv (H, \mathcal{K}(H))$. Since there Hilbert spaces are λ-categorical for $\lambda > \aleph_0$, $Th(H, \mathcal{K}(H))$ is λ-categorical. In the same way as previous theorem, recall that all of these representations are degenerate and $\mathcal{K}(H)$ is not unital. □

Theorem 3.9.3. *There is a prime model for $Th(H, \mathcal{K}(H))$.*

Proof. We have that $H = H_d$ and $H_e = 0$. We know that in any model $(H', \pi') \models Th(H, \pi)$, $H_d \subseteq H$, so (H, π) is a prime model for $Th(H, \pi)$. □

3.9.2 The structure $(H, B(H))$

In this case, $H_d = H$. If $(H', \pi') \models Th(H, B(H))$, then $H' = H \oplus G$, where $(G, \pi' \restriction G)$ is a generic representation of the Calkin algebra $\mathcal{C}(H)$.

Remark 3.9.4. This case is similar to the case of (H, N), with $\sigma_d(N) = \emptyset$.

3.9.3 The structure (H, N) where N is a bounded normal operator on H

Let H be a separable Hilbert space and let N be a bounded normal operator on H. Without loss of generality we can assume that $\|N\| \leq \frac{1}{2}$.

Lemma 3.9.5. *The operator N^* is definable in $< H, 0, +, \langle \mid \rangle, N >$*

Proof. Let $P(x,y) = \sup_z |\langle Nz|x\rangle - \langle z|y\rangle|$. We have that $\|N^*x-y\|^2 = |\langle N^*x-y|N^*x-y\rangle| \leq \sup_z |\langle z|N^*x-y\rangle| = \sup_z |\langle z|N^*x\rangle - \langle z|y\rangle| = P(x,y)$. By Proposition 9.19 in and Definition 9.22 in [21], N^* is definable □

Theorem 3.9.6. *Let $f \in \mathcal{B}(\sigma(N),\mathbb{C})$. Then $f(N)$ is $\emptyset$-definable if and only if $f \in \mathcal{C}(\sigma(N),\mathbb{C})$.*

Proof. ⇒) Suppose $f \in \mathcal{B}(\sigma(N),\mathbb{C}) \setminus \mathcal{C}(\sigma(N),\mathbb{C})$ is such that $f(N)$ is definable. Let λ_0 be a point of discontinuity of f. Let $(\lambda_k)_{k\in\mathbb{N}}$ a sequence in $\sigma(N)$ and $\mathcal{U}$ an ultrafilter over $\mathbb{N}$ such that $\lim_{\mathcal{U}} \lambda_k = \lambda_0$, $\lim_{\mathcal{U}} f(\lambda_k)$ exists but $\lim_U f(\lambda_k) \neq f(\lambda_0)$. There exist models $\mathcal{H}_k$ and a unitary $v_k \in \mathcal{H}_k$ such that $\mathcal{H}_k \models Nv_k - \lambda_k v_k = 0$. Let $\mathcal{H} = \Pi_{\mathcal{U}}\mathcal{H}_k$ and let $v = (v_k)_{\mathcal{U}} \in \mathcal{H}$. Then $(v_k)_{\mathcal{U}}$ is an eigenvector in $\mathcal{H}$ for the eigenvalue λ_0 and we have:

$$\begin{aligned} f(\lambda_0)v = f(N)(v) = f(N)(v_k)_{\mathcal{U}} = (f(N)v_k)_{\mathcal{U}} = \\ = (f(\lambda_k)v_k)_{\mathcal{U}} = (\lim_U f(\lambda_k))(v_k)_{\mathcal{U}} = (\lim_U f(\lambda_k))v \end{aligned}$$

So $f(\lambda_0) = \lim_{\mathcal{U}} f(\lambda_k)$ contradiction.

⇐) Suppose $f \in \mathcal{C}(\sigma(N),\mathbb{C})$. Then by Stone-Wierstrass theorem f can be uniformly aproximated by a sequence of polynomials in z and $\bar{z}$ over $\sigma(N)$.These polynomials are tranlated into polynomials in N and N^*. By Lemma 3.9.5, such polyinomials are definable, so $f(N)$ is definable. □

Proposition 3.9.7. *An automorphism U of (H,N) is an unitary operator of $\mathcal{H}$ such that $UN = NU$.*

Proof. It is clear that U must be a linear operator. Also, we have that for every $u, v \in \mathcal{H}$ we must have that $U(Nv) = N(Uv)$ and $\langle Uu \,|\, Uv\rangle = \langle u \,|\, v\rangle$ by definition of automorphism. Therefore U must be unitary and commute with N. □

Remark 3.9.8. If T is an $\emptyset$-definable operator on (H,N) and U is an automorphism of (H,N), then $TU = UT$ by $\emptyset$-definability.

Remark 3.9.9. We denote by $\tilde{\mathcal{H}} =< \tilde{H}, \tilde{N} >$ an elementary extension of (H,N) which is saturated and homogeneous.

Lemma 3.9.10. *Let T be an $\emptyset$-definable operator in $\tilde{\mathcal{H}}$. Then $T \restriction \tilde{\mathcal{H}}_\lambda = \alpha_\lambda Id_{\tilde{\mathcal{H}}_\lambda}$ for some $\alpha_\lambda \in \mathbb{C}$, where $\tilde{\mathcal{H}}_\lambda$ is the eigenspace in $\tilde{\mathcal{H}}$ corresponding to the spectral value λ.*

Proof. Let $\lambda \in \sigma(N)$ and U be an automorphism of $\tilde{\mathcal{H}}_\lambda$. We have that $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_\lambda \oplus \tilde{\mathcal{H}}_\lambda^\perp$. Let $\tilde{U} = U \oplus Id_{\tilde{\mathcal{H}}_\lambda^\perp} \in \mathrm{Aut}(\tilde{\mathcal{H}}_\lambda)$. By Remark 3.9.8, T commutes with $\tilde{U}$, and therefore, $T \restriction \mathcal{H}_\lambda : \tilde{\mathcal{H}}_\lambda \to \tilde{\mathcal{H}}_\lambda$ commutes with U. Then $T \restriction \tilde{\mathcal{H}}_\lambda$ commutes with every automorphism of $\tilde{\mathcal{H}}_\lambda$, and by Schur's lemma, there exists a complex number α_λ such that $T \restriction \tilde{\mathcal{H}}_\lambda = \alpha_\lambda Id_{\tilde{H}_\lambda}$ □

Definition 3.9.11. The *C^*-algebra generated by N in $B(H)$* is the least C^* subalgebra of $B(H)$ that contains N and we denote it by $C^*(N)$.

Theorem 3.9.12. *The $\emptyset$-definable operators over (H,N) are exactly the ones in $C^*(N)$.*

Proof. By Theorem 3.9.6, all the elements in $C^*(N)$ are definable.
Conversely, let $(\tilde{H},\tilde{N})$ be as above, and let T be $\emptyset$-definable. By previous Lemma, given $\lambda \in \sigma(N)$, there exists $\alpha_\lambda \in \mathbb{C}$ such that $T \restriction \tilde{\mathcal{H}}_\lambda = \alpha_\lambda Id_{\tilde{\mathcal{H}}_\lambda}$. Let $f : \sigma(N) \to \mathbb{C}$ be defined by $f(\lambda) = \alpha_\lambda$. By the $\Leftarrow$ part of the proof of Theorem 3.9.6 f must be continuous. On the other hand,$(f(N) - T) \restriction \tilde{\mathcal{H}}_\lambda \equiv 0$ for all $\lambda \in \sigma(N)$, so we have that $(f(N) - T) \equiv 0$, so $f(N) = T$. Then, $T \in C^*(N)$. □

The following is a straightforward conclusion this theorem, which is similar to the von Neumann bicommutant theorem:

Corollary 3.9.13. *The C^*-algebra defined by N is exactly the set of operators that commute with all the unitary operators that commute with N.*

Let $\sigma_d(N)$ and $\sigma_e(N)$ be the discrete and the essential spectrum of N (see Definition .4.27). Since all elements in $\sigma_d(N)$ are isolated, any function f on $\sigma_d(N)$ is continuous and therefore $f(N)$ is definable. Since for every $\lambda \in \sigma_d(N)$ the eigenspace for λ is finite dimensional, say n_λ, we get that:

$$H_d \simeq \bigoplus_{\lambda \in \sigma_d(N)} \mathbb{C}^{n_\lambda}$$

On the other hand, we have that

$$H_e \simeq \bigoplus_{\mu \in M} L^2(\sigma_e(N), \mu)$$

where M is the set of Borel measures $\sigma_e(N)$ realized in H. So we get the following results:

Theorem 3.9.14. *If $\sigma_e(N) = \emptyset$, then $Th(H,N)$ has only one model.*

Proof. If $\sigma_e(N) = \emptyset$ (see Definition .4.11), then $\sigma(N) = \sigma_d(N)$ and every $\lambda \in \sigma(N)$ is an eigenvalue with finite dimensional dimension. Also, $\sigma(N)$ is finite since, otherwise, there would be an essential spectral value that would be an accumulation point of $\sigma(N)$. So $H = H_d$ is finite dimensional. For any $(H', \mathbb{N}') \models Th(H,N)$, $\sigma(N) = \sigma_d(N) = \sigma_d(N') = \sigma(N')$ and for every $\lambda \in \sigma(N')$ the dimension of the eigenspace corresponding to λ in (H,N) is the same as its dimension in (H',N') so $(H,N) \equiv (H',N')$. □

Theorem 3.9.15. *If $\sigma_e(N) = \{0\}$, and 0 is an isolated point in $\sigma(N)$, then $Th(H,N)$ is separably categorical.*

Proof. If 0 is an isolated point in $\sigma(N)$ and $0 \in \sigma_e(N)$, it means that the dimension of the eigenspace corresponding to 0 is $\aleph_0$. So, any other model of $Th(H,N)$ must have the same dimension in the eigenspace corresponding to 0 and, therefore, it must be isometrically isomorphic to (H,N). □

Theorem 3.9.16. *If $\sigma_e(N) = \{0\}$, and 0 is an accumulation point of elements in $\sigma_d(N)$, then there are countably many models of $Th(H,N)$ with density $\aleph_0$.*

Proof. If $\sigma_e(N) = \{0\}$ and 0 is an accumulation point of elements in $\sigma_d(N)$, let G is any separable Hilbert space and let $N_G \equiv 0$ on G. Let $H' = H_d \oplus G$ and let $N' := N_d \oplus N_G$. Then, $(H', \pi') \equiv (H, \pi)$. Since there are countably many separable Hilbert spaces, one for each possible dimension $n = 1, 2, \dots, \aleph_0$, $Th(H,N)$ has countably many models with density $\aleph_0$. □

Theorem 3.9.17. *If $\sigma_e(N) = \{0\}$, then $Th(H,N)$ is λ-categorical for $\lambda > \aleph_0$.*

Proof. Similarly to the proof of previous theorem, let G is any Hilbert space with density $> \aleph_0$. For $a \in \mathcal{A}$ and $v \in G$, let $N_G \equiv 0$ on G. Let $H' = H \oplus G$ and let $\pi' := \pi \oplus \pi_G$. Then, $(H', \pi') \equiv (H, \pi)$. Since there Hilbert spaces are λ-categorical for $\lambda > \aleph_0$, $Th(H,N)$ is λ-categorical. □

Theorem 3.9.18. *If $\sigma_e(N) = \{0\}$, and 0 is an accumulation point of elements in $\sigma_d(N)$, then there is a prime model for $Th(H,\pi)$.*

Proof. Take then $H' = H_d$ and let $N' := N \restriction H_d$. Then $(H', N') \models Th(H,N)$ and (H', N') is a substructure of any model of Th(H,N) . □

Theorem 3.9.19. *If $dens(\sigma_e(N)) > \aleph_0$, then there are uncountably many separable non-isomorphic models of $Th(H,N)$*

Proof. We have that

$$H_e \simeq \bigoplus_{\mu \in M} L^2(\sigma_e(N), \mu)$$

where M is the set of Borel measures $\sigma_e(N)$ realized in H. So, for every set of measures $\{\mu_i \mid i \in I\}$ such that $\cup_{i\in I} supp(\mu_i) = \sigma_e(N)$, there is an isomorphism class of models of $Th(H,N)$. Since there are uncountably many such families of Borel sets in $\sigma_e(N)$, there are uncontably many separable non-isomorphic models of $Th(H,N)$. □

Remark 3.9.20. The isolated types are those that correspond to Borel measures whose support is a subset of $\sigma_d(N)$. So, if $supp(\mu) \not\subseteq \sigma_d(N)$, then there is a model of $Th(H,N)$ that omits μ.

Theorem 3.9.21. *If $\sigma_d(N) = \emptyset$, then there is no prime model for $Th(H,\pi)$.*

Proof. If $\sigma_d(N) = \emptyset$, then $H = H_e$. So, for any Borel measure μ such that $supp(\mu) = \sigma_e(N)$ then $L^2(\sigma_e(N), \mu)$ is model of $Th(H, \pi)$. □

Theorem 3.9.22. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\mu_v \perp \mu_w$.*

Proof. $p \perp^a q$ if and only if $\mathcal{H}_{v'} \perp \mathcal{H}_{w'}$ for all $v' \models p$ and $w' \models q$. By Lesbesgue decomposition theorem $\mu_w = \mu_v^{\|} + \mu_v^{\perp}$ where, $\mu_v^{\|} << \mu_v$ and $\mu_v^{\perp} \perp \mu_v$. $\mu_v^{\|} \neq 0$ if and only if there is a choice of $v' \models p$ and $w' \models q$ such that $\mathcal{H}_{v'} \cap \mathcal{H}_{w'} \neq \{0\}$ and therefore $\mathcal{H}_{v'} \not\perp \mathcal{H}_{w'}$. □

Corollary 3.9.23. *Let $A \subseteq H$ be such that $A = acl(A)$. Let $p, q \in S_1(A)$, let $v \models p$ and $w \models q$. Then, $p \perp^a_A q$ if and only if $\mu_{P^{\perp}_{acl(A)}(v)} \perp \mu_{P^{\perp}_{acl(A)}(w)}$*

Proof. Clear from previous theorem. □

Corollary 3.9.24. *Let $A \subseteq H$ be such that $A = acl(A)$. Let $p, q \in S_1(A)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Theorem 3.9.25. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \rhd_\emptyset q$ if and only if $\mu_v >> \mu_w$.*

Proof. Suppose $p \rhd_\emptyset q$. Suppose that v and w are such that if $v \mathop{\smile\!\!\!\!\!\!\;|}^*_\emptyset A$ then $w \mathop{\smile\!\!\!\!\!\!\;|}^*_\emptyset A$ for every A. Then for every A if $\mathcal{H}_v \perp \mathcal{H}_A$ then $\mathcal{H}_w \perp \mathcal{H}_A$. This means $\mathcal{H}_w \subseteq \mathcal{H}_v$ and $\mathcal{H}_w$ is unitarily equivalent to some Hilbert subspace of $\mathcal{H}_w$ and by Theorem 3.8.1 $\mu_w << \mu_v$. □

Corollary 3.9.26. *Let A, $\mathfrak{B}$ be small subsets of $\tilde{H}$ and $p \in S_1(A)$ and $q \in S_1(B)$ two stationary types. Then $p \rhd_C q$ if and only if there exist v $w \in \tilde{\mathcal{H}}$ such that $tp(v/C)$ is a non-forking extension of p, $tp(w/C)$ is a non-forking extension of q and $\mu_{P^{\perp}_{acl(A)}v} >> \mu_{P^{\perp}_{acl(A)}w}$.*

Proof. Clear from previous theorem. □

3.9.4 A generic representation of an abelian C^*-algebra

Let $\mathcal{A}$ be an abelian C^*algebra. By Gelfand Representation Theorem there exists a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically is isomorphic to $C_0(X)$, the completion of the space of complex function on X which are 0 outside a compact subset of X.

According to [2], in a generic representation of $\mathcal{A}$, the space of types over the emptyset of vectors with norm 1 is homeomorphic to the space of states of $\mathcal{A}$ under the weak* topology. By, Riesz Representation Theorem, this is homeomorphic to the space of positive Borel measures over X with the weak* topology.

3.9.5 A generic representation of $C^*(S_3)$

An example of a group C^*-algebra (see [27]) is $C^*(S_3)$. Since S_3 is a finite (therefore discrete) group with 6 elements, $C^*(S_3)$ must be 6-dimensional. Since S_3 is not abelian, $C^*(S_3)$ must be non-commutative. The only (up to isomorphism) 6-dimensional non-commutative C^*-algebra is $\mathbb{C} \oplus \mathbb{C} \oplus M_{2\times 2}$. So, $C^*(S_3) = \mathbb{C} \oplus \mathbb{C} \oplus M_{2\times 2}$ (see [27]). This algebra acts on $\mathbb{C}^4$. This is a finite (algebraic) representation so, as in the case of a normal operator N with $\sigma_e(N) = \emptyset$, $(\mathbb{C}^4, C^*(S_3))$ is totally categorical.
Let $\tilde{H} := \bigoplus_\omega \mathbb{C}^4$. For every $a \in \bigoplus_\omega \mathbb{C} \oplus \mathbb{C} \oplus M_{2\times 2}$ and for every $(\bar{v_k})_{\kappa\in\mathbb{N}} \in \tilde{H}$, let $\tilde{\pi}(a) := (a(\bar{v_k}))$. No element in $C^*(S_3)$ acts as a compact operator in $\tilde{H}$, so $(\tilde{H}, \tilde{\pi})$ is a generic representation of $C^*(S_3)$.
For $(\bar{v_k})_{\kappa\in\mathbb{N}} \in \tilde{H}$, there are $(v_k^i)_{\kappa\in\mathbb{N},k=1,2,3,4} \in \mathbb{C}$ so that for all $k \in \mathbb{N}$, $\bar{v_k} = (v_k^1, v_k^2, v_k^3, v_k^4)$. The state on $C^*(S_3)$ defined by $\bar{v} = (\bar{v_k})_{\kappa\in\mathbb{N}} \in \tilde{H}$ is $\phi_{\bar{v}}(\cdot) = \sum_k \phi_{v_k^1} + \sum_k \phi_{v_k^2} + \sum_k \phi_{(v_k^3, v_k^4)}$. So the type over the emptyset of $\bar{v}$ is determined by the triplet $(\sum_k \phi_{v_k^1}, \sum_k \phi_{v_k^2}, \sum_k \phi_{(v_k^3, v_k^4)})$.

4 Model theory of a Hilbert space expanded with an unbounded closed selfadjoint operator

4.1 Introduction

This chapter deals with a complex Hilbert space expanded by an unbounded closed self-adjoint operator Q, from the point of view of *Metric Abstract Elementary Classes* (see [18], [19] and [35]).
The main results in this chapter are the following:

- We build a Metric Abstract Elementary Class associated with the Hilbert space H and the operator Q which is denoted by $\mathcal{K}_{(H,\Gamma_Q)}$, where Γ_Q stands for the distance to the graph of the operator Q (see Theorem 4.2.12).

- We characterize (Galois) types of vectors in some structure in $\mathcal{K}_{(H,\Gamma_Q)}$, in terms of spectral measures (see Theorem 4.2.21).

- We show that $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-categorical and $\aleph_0$-stable up to a system of perturbations (see Theorem 4.4.5).

- We characterize continuous first order elementary equivalence of structures of the type (H, Γ_Q) (see Theorem 4.5.9).

- We give a continuous $L_{\omega_1,\omega}$ axiomatization of the class $\mathcal{K}_{(H,\Gamma_Q)}$ (see Theorem 4.5.11).

- We characterize non-splitting in $\mathcal{K}_{(H,\Gamma_Q)}$ and we show that it has the same properties as non-forking for superstable first order theories (see Theorem 4.6.15).

This chapter is divided as follows: In Section 4.2, we define a *metric abstract elementary class* associated with (H, Γ_Q) (denoted by $\mathcal{K}_{(H,\Gamma_Q)}$). In Section 4.3, we give a characterization of definable and algebraic closures. In Section 4.4, we define a system of perturbations for $\mathcal{K}_{(H,\Gamma_Q)}$, and show that the class is $\aleph_0$-categorical up to the (previously defined) system of perturbations. In Section 4.5, we give a characterization of first order elementary equivalence and give a continuous $L_{\omega_1,\omega}$ axiomatization of the class $\mathcal{K}_{(H,\Gamma_Q)}$. In Section 4.6, we define

spectral independence in $\mathcal{K}_{(H,\Gamma_Q)}$ and we show that it is equivalent to non-splitting and has the same properties as non-forking for superstable first order theories. Finally in Section 4.7, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures.
The theoretical preliminaries about Spectral Theory of unbounded closed selfadjoint operators, needed in the rest of the chapter are in Section .4.

4.2 A metric abstract elementary class defined by (H, Γ_Q)

In this section we define a *metric abstract elementary class* associated with a closed unbounded self-adjoint operator Q defined on a Hilbert space (see Definition 4.2.9). We will recall several notions related with metric abstract elementary classes that come from [35].

Definition 4.2.1. An $\mathcal{L}$-*metric structure* $\mathcal{M}$, for a fixed similarity type $\mathcal{L}$, consists of:

- A closed metric space (M, d)
- A family $(R^{\mathcal{M}})_{R\in\mathcal{L}}$ of continuous functions from M^{n_R} into $\mathbb{R}$, where n_R is the arity of R.
- An indexed family $(F^{\mathcal{M}})_{F\in\mathcal{L}}$ of continuous functions on powers of M.
- An indexed family $(c^{\mathcal{M}})_{c\in\mathcal{L}}$ of distinguished elements of M.

We write this structure as

$\mathcal{M} = (M, d, (R^{\mathcal{M}})_{R\in\mathcal{L}}, (F^{\mathcal{M}})_{F\in\mathcal{L}}, (c^{\mathcal{M}})_{c\in\mathcal{L}})$.

If $\mathcal{M}$ is a metric structure, $dens(\mathcal{M})$ denotes the smallest cardinal of a dense subset of M.

Definition 4.2.2. Let $\mathcal{L} = (0, -, i, +, (I_r)_{r\in\mathbb{Q}}, \|\cdot\|, \Gamma_Q)$. A *Hilbert space operator* structure for $\mathcal{L}$ is a metric structure of only one sort:

$(H, 0, +, i, (I_r)_{r\in\mathbb{Q}}, \mathbb{R}, \|\cdot\|, \Gamma_Q)$

where

- H is a Hilbert space
- Q is a closed (unbounded) self-adjoint operator on H (see Definition .4.5 and Definition .4.9).
- 0 is the zero vector in H
- $+ : H \times H \to H$ is the usual sum of vectors in H

- $i : H \to H$ is the function that to any vector $v \in H$ assigns the vector iv where $i^2 = -1$
- $I_r : H \to H$ is the function that sends every vector $v \in H$ to rv, where $r \in \mathbb{Q}$
- $\mathbb{R}$ is the sort of reals
- $\| \cdot \| : H \to \mathbb{R}$ is the norm function
- $\Gamma_Q : H \times H \to \mathbb{R}$ is the function assigning to each v, $w \in H$ the number $\Gamma_Q(v, w)$, which is the distance of (v, w) to the graph of Q. Since Q is closed, $\Gamma_Q(v, w) = 0$ if and only if (v, w) belongs to the graph of Q.

Briefly, the structure will be refered to either as (H, Γ_Q). (H, Γ_Q) is a metric structure for the similarity type $\mathcal{L}$.

Definition 4.2.3. Let Q be a closed unbounded self-adjoint operator on a Hilbert space H. For $\lambda \in \sigma_d(Q)$, let n_λ be the dimension of the eigenspace corresponding to λ. We define the *discrete part* of H in the following way:

$$H_d := \bigoplus_{\lambda \in \sigma_d(Q)} \mathbb{C}^{n_\lambda}$$

In the same way, we define $Q_d := Q \restriction H_d$

Definition 4.2.4. Let Q be a closed unbounded self-adjoint operator on a Hilbert space H. We define the *essential part* of H in the following way:

$$H_e := H_d^\perp$$

In the same way, we define $Q_e := Q \restriction H_e$

Definition 4.2.5. Given $G \subseteq H$ and $v \in H$, we denote by:

1. H_G, the Hilbert subspace of H generated by the elements $h(Q)v$, where $v \in G$, h is a bounded Borel function on $\mathbb{R}$ and $v \in D(h(Q))$.
2. $Q_G := Q \restriction H_G$.
3. H_v, the space H_G when $G = \{v\}$ for some vector $v \in H$
4. $Q_v := Q_G$ when $G = \{v\}$.
5. $H_G^\perp$, the orthogonal complement of H_G
6. P_G, the projection over H_G.
7. $P_{G^\perp}$, the projection over $H_G^\perp$.

Definition 4.2.6. Given $G \subseteq H$ and $v \in H$, we denote by $(H_G)_d$ and $(H_G)_e$ the projections of H_G on H_d and H_e respectively.

Corollary 4.2.7. *There is a set $G \subseteq H$ such that $H = H_d \oplus \bigoplus_{v \in G} H_v$. Further $G \subseteq H_e$.*

Proof. Apply Fact .4.38 on H_d and H_e separately. ☐

Lemma 4.2.8. *Let Q_1 and Q_2 be closed unbounded self-adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. An isomorphism $U : (H_1, \Gamma_{Q_1}) \to (H_2, \Gamma_{Q_2})$ is a unitary operator of $U : H_1 \to H_2$ such that $UD(Q_1) = D(Q_2)$ and $UQ_1 v = Q_2 U v$ for every $v \in D(Q_1)$.*

Proof. $\Rightarrow$ Suppose U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). It is clear that U must be a linear operator. Also, we have that for every u, $v \in \mathcal{H}$ we must have that $\langle Uu \,|\, Uv \rangle = \langle u \,|\, v \rangle$ by definition of automorphism. Therefore U must be an isometry and, therefore, it must be unitary.

Furthermor, as U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}), for every $(v, w) \in H \times H$ we have that $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Therefore, $\Gamma_{Q_1}(v, w) = 0$ if and only if $\Gamma_{Q_2}(Uv, Uw) = 0$. So, for every $v \in D(Q_1)$, $UQ_1 v = Q_2 U v$.

$\Leftarrow$ Let $U : H_1 \to H_2$ be an unitary operator such that $UD(Q_1) = D(Q_2)$ and $UQ_1 v = Q_2 U v$ for every $v \in D(Q_1)$. It remains to show that for every $(v, w) \in H \times H$, $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Let $(v, w) \in H \times H$ be any pair of vectors. There exists a sequence of pairs $(v_n, w_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $v_n \in D(Q_1)$, $w_n = Q_1 v_n$ and $\Gamma_{Q_1}(v, w) = \lim_{n \to \infty} d[(v, w); (v_n, w_n)]$.

By hypothesis, U is an isometry, and maps the graph of Q_1 into the graph of Q_2; so for all $n \in \mathbb{N}$, $Uv_n \in D(Q_2)$ and $Uw_n = Q_2 v_n$. We have that

$$\lim_{n \to \infty} d[(Uv, Uw); (Uv_n, Uw_n)] = \lim_{n \to \infty} d[(v, w); (v_n, w_n)] = \Gamma_{Q_1}(v, w).$$

So $\Gamma_{Q_2}(Uv, Uw) \leq \Gamma_{Q_1}(v, w)$. Repeating the argument for U^{-1}, we get $\Gamma_{Q_1}(v, w) \leq \Gamma_{Q_2}(Uv, Uw)$.

☐

Definition 4.2.9. A *Metric Abstract Elementary Class* (MAEC), on a fixed similarity type $\mathcal{L}(\mathcal{K})$, is a class $\mathcal{K}$ of $\mathcal{L}(\mathcal{K})$-metric structures provided with a partial order $\prec_{\mathcal{K}}$ such that:

1. Closure under isomorphism:
   a) For every $\mathcal{M} \in \mathcal{K}$ and every $\mathcal{L}(\mathcal{K})$-structure $\mathcal{N}$, if $\mathcal{M} \simeq \mathcal{N}$ then $\mathcal{N} \in \mathcal{K}$.
   b) Let $\mathcal{N}_1$, $\mathcal{N}_2 \in \mathcal{K}$ and $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$ be such that there exists $f_l : \mathcal{N}_l \simeq \mathcal{M}_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$. Then $\mathcal{N}_1 \prec_{\mathcal{K}} \mathcal{N}_2$ implies that $\mathcal{M}_1 \prec_{\mathcal{K}} \mathcal{M}_2$.
2. For all $\mathcal{M}$, $\mathcal{N} \in \mathcal{K}$ if $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$.

3. Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}^*$ be $\mathcal{L}(\mathcal{K})$-structures. If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}^*$ and $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}^*$, then $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$.

4. Downward Löwenheim-Skolem: There exists a cardinal $LS(\mathcal{K}) \geq \aleph_0 + |\mathcal{L}(\mathcal{K})|$ such that for every $\mathcal{M} \in \mathcal{K}$ and for every $A \subseteq M$ there exists $\mathcal{N} \in \mathcal{K}$ such that $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$, $N \supseteq A$ and $dens(N) \leq |A| + LS(\mathcal{K})$.

5. Tarski-Vaught chain:

   a) For every cardinal μ and every $\mathcal{N} \in \mathcal{K}$, if $\{\mathcal{M}_i \prec_{\mathcal{K}} \mathcal{N} \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing and continuous (i.e. $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) then $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \in \mathcal{K}$ and $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \prec_{\mathcal{K}} \mathcal{N}$.

   b) For every μ, if $\{\mathcal{M}_i \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) and continuous then $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \in \mathcal{K}$ and for every $j < \mu$, $\mathcal{M}_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\mu} \mathcal{M}_i}$.

   Here, $\overline{\bigcup_{i<\mu} \mathcal{M}_i}$ denotes the completion of $\bigcup_{i<\mu} \mathcal{M}_i$.

Definition 4.2.10. Let (H, Γ_Q) be a structure as described in Definition 4.2.2. Let $\mathcal{L}$ the similarity type of (H, Γ_Q). We define $\mathcal{K}_{(H,\Gamma_Q)}$ to be the following class:

$\mathcal{K}_{(H,\Gamma_Q)} := \{(H', \Gamma_{Q'}) \mid (H', \Gamma_{Q'})$ is an $\mathcal{L}$ Hilbert space operator structure and $Q' \sim_\sigma Q\}$

We define the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,\Gamma_Q)}$ by:

$(H_1, \Gamma_{Q_1}) \prec_{\mathcal{K}} (H_2, \Gamma_{Q_2})$ if and only if $H_1 \subseteq H_2$ and $Q_1 \subseteq Q_2$

Remark 4.2.11. $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,\Gamma_Q)}$ is trivial i.e. coincides with $\subseteq$

Theorem 4.2.12. *The class $\mathcal{K}_{(H,\Gamma_Q)}$ is a MAEC.*

Proof. 1. Closure under isomorphism:

   a) Let $(H_1, \Gamma_{Q_1}) \simeq (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$. This means that $Q_1 \sim_\sigma Q$. Since isomorphisms preserve spectral properties, $Q_2 \sim_\sigma Q$ and $(H_1, \Gamma_{Q_1}) \in \mathcal{K}_{(H,\Gamma_Q)}$.

   b) Clear since by Remark 4.2.11 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,\Gamma_Q)}$.

2. Clear since by Remark 4.2.11 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,\Gamma_Q)}$.

3. Clear since by Remark 4.2.11 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,\Gamma_Q)}$.

4. $LS(\mathcal{K}) \leq 2^{2^{\aleph_0}}$. We first prove the following claim:

   Claim. If $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H,\Gamma_Q)}$, there is a $(H'', \Gamma_{Q''}) \prec_{\mathcal{K}} (H', \Gamma_{Q'})$ such that $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $Dens(H'') \leq 2^{2^{\aleph_0}}$.

Proof. By Corollary 4.2.7, there is a set $G' \subseteq H'$ such that $H' = H_d \oplus \bigoplus_{v \in G'} H'_v$ (see Definition .4.11 and Definition 4.2.5). Since there are at most $2^{2^{\aleph_0}}$ many Borel measures, there is a $G'' \subseteq G'$ such that $|G''| \leq 2^{2^{\aleph_0}}$ and for every $v \in G'$ there is a $w \in G''$ such that $\mu_v = \mu_w$ (see Definition .4.36). On the other hand, since $\sigma_D(Q)$ is at most countable, $Dens(H_d) \leq 2^{\aleph_0}$.

Take

$$H'' = H_d \oplus \bigoplus_{v \in G''} H'_v$$

and

$$Q'' := Q' \restriction H''$$

We have that H'' is a closed subset of H'; thas, the graph of Q'' is also closed and therefore Q'' is a closed operator. Since $G' \subseteq H_e$, and $H''_d = H_d$, we get that $Q'' \sim_\sigma Q$. Then $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $(H'', \Gamma_{Q''}) \prec_{\mathcal{K}} (H', \Gamma_{Q'})$ and $|H''| \leq 2^{2^{\aleph_0}}$. □

Now, let $(H', \Gamma_{Q'}) \in \mathcal{K}$ and $A \subseteq H'$. Let G' be as in Corollary 4.2.7 and let $(H'', \Gamma_{Q''})$ be as in the previous Claim. Since $A \subseteq H_d \oplus \bigoplus_{v \in G''} H'_v$, there is a $G_A \subseteq G''$, with $|G_A| \leq |A| \cdot \aleph_0$, such that $A \subseteq H_d \oplus \bigoplus_{v \in G_A} H'_v$.

Let

$$\hat{H} := H_d \oplus \bigoplus_{v \in G_A \cup G''} H'_v$$

and

$$Q'' := Q' \restriction \hat{H}$$

We have that Q'' is closed since $\hat{H}$ is a closed subset of H' and so is the graph of Q''. Then $(\hat{H}, \Gamma_{\hat{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $(\hat{H}, \Gamma_{\hat{Q}}) \prec_{\mathcal{K}} (H', \Gamma_{Q'})$, $A \subseteq \hat{H}$ and $Dens(\hat{H}) \leq |A| + 2^{2^{\aleph_0}}$.

5. Tarski-Vaught chain:
   a) Suppose κ is a regular cardinal and $(\hat{H}, \Gamma_{\hat{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$. Let $(H_i, \Gamma_{Q_i})_{i<\kappa}$ a $\prec_{\mathcal{K}}$ increasing sequence such that $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$ for all $i < \kappa$. Then, for all $i < \kappa$ $(H_{i+1}, \Gamma_{Q_{i+1}}) = (H_i, \Gamma_{Q_i}) \oplus (H'_i, \Gamma_{Q'_i})$, where H'_i is a Hilbert space and Q'_i is a (possibly unbounded) closed selfadoint operator such that $\sigma_d(Q'_i) = \emptyset$ and $\sigma_e(Q'_i) \subseteq \sigma_e(\hat{Q})$. Then $\overline{\bigcup_{i<\kappa}(H_i, \Gamma_{Q_i})} = (H_0, \Gamma_{Q_0}) \bigoplus_{i<\kappa}(H'_i, \Gamma_{Q'_i})$.

      Let $(\bar{H}, \Gamma_{\bar{Q}}) := \overline{\bigcup_{i<\kappa}(H_i, \Gamma_{Q_i})}$. We have then that $(\bar{H}, \Gamma_{\bar{Q}}) = (H_0, \Gamma_{Q_0}) \bigoplus_{i<\kappa}(H'_i, \Gamma_{Q'_i})$. Since $H_0 \perp H'_i$ for every $i < \kappa$ and $H'_i \perp H'_j$ for all $i \neq j < \kappa$, we have that $\sigma_d(\bar{Q}) = \sigma_d(Q_0) \cup \bigcup_{i<\kappa} \sigma_d(Q'_i)$. $\sigma_d(Q') = \emptyset$ $\sigma_e(Q') \subseteq \sigma_e(\hat{Q}) = \sigma_e(Q_0)$ and so,

$\sigma_d(\bar{Q}) = \sigma_e(Q_0)$ and $\sigma_e(\bar{Q}) = \sigma_e(Q_0)$. In the same way, for $\lambda \in \sigma(\bar{Q})$, $\bar{H}_\lambda = (H_0)_\lambda$. This proves that $Q_0 \sim_e Q'$ and, therefore, $(\bar{H}, \Gamma_{\bar{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$.

Since $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$ for every $i < \kappa$, $\overline{\bigcup_{i<\kappa}(H_i, \Gamma_{Q_i})} \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$.

b) In previous construction, it is clear that $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\bar{H}, \Gamma_{\bar{Q}}) = \overline{\bigcup_{i<\kappa}(H_i, \Gamma_{Q_i})}$ for every $i < \kappa$.

□

Remark 4.2.13. From now on, the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,\Gamma_Q)}$ will be denoted as $\prec$.

Definition 4.2.14. Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be a MAEC and let $\mathcal{M}$, $\mathcal{N} \in \mathcal{K}$ be two structures. An embedding $f : \mathcal{M} \to \mathcal{N}$ such that $f(\mathcal{M}) \prec_{\mathcal{K}} \mathcal{N}$ is called a $\mathcal{K}$-*embedding*.

Definition 4.2.15. A MAEC $\mathcal{K}$ has the *Joint Embedding Property* (JEP) if for any $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$ there are $\mathcal{N} \in \mathcal{K}$ and a $\mathcal{K}$-embeddings $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$.

Theorem 4.2.16. *$\mathcal{K}_{(H,\Gamma_Q)}$ has the JEP.*

Proof. Let (H_1, Γ_{Q_1}), $(H_1, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$. Without loss of generality, we can assume that $H_1 \cap H_2 = \emptyset$. By Corollary 4.2.7, there are sets $G_1 \subseteq (H_1)_e$ and $G_2 \subseteq (H_2)_e$ such that $H_1 = H_d \oplus \bigoplus_{v\in G_1}(H_1)_v$ and $H_2 = H_d \oplus \bigoplus_{v\in G_2}(H_2)_v$.
Let

$$\hat{H} = H_d \oplus \bigoplus_{v\in G_1}(H_1)_v \oplus \bigoplus_{v\in G_2}(H_2)_v$$

and

$$\hat{Q} := (Q_1 \restriction H_d) \oplus \Big(\bigoplus_{v\in G_1}(Q_1 \restriction (H_1)_v)\Big) \oplus \Big(\bigoplus_{v\in G_2}(Q_2 \restriction (H_2)_v)\Big)$$

We have that $\hat{H}_d \simeq H_d$. Since $H_1 = H_d \oplus \bigoplus_{v\in G_1}(H_1)_v$ and $G_1 \subseteq (H_1)_e$, we have that $(H_1)_e = \bigoplus_{v\in G_1}(H_1)_v$. In the same way $(H_1)_e = \bigoplus_{v\in G_2}(H_2)_v$. So, $\sigma(Q_1 \restriction H_d) = \sigma_d(Q)$, $\sigma\big(\bigoplus_{v\in G_1}(Q_1 \restriction (H_1)_v)\big) = \sigma\big(\bigoplus_{v\in G_2}(Q_2 \restriction (H_2)_v)\big) = \sigma_e(Q)$. Therefore, $\hat{Q} \sim_\sigma Q$, $(\hat{H}, \Gamma_{\hat{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $Id_{H_d} \oplus \bigoplus_{v\in G_1} Id_{(H_1)_v}$ and $Id_{H_d} \oplus \bigoplus_{v\in G_2} Id_{(H_2)_v}$ are respective $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings from (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) to $(\hat{H}, \Gamma_{\hat{Q}})$. □

Definition 4.2.17. A MAEC $\mathcal{K}$ has the *Amalgamation Property* (AP) if for any $\mathcal{M}$, $\mathcal{N}_1$, $\mathcal{N}_2 \in \mathcal{K}$ such that $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_1$ and $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_2$, there are $\mathcal{M}' \in \mathcal{K}$ and a $\mathcal{K}$-embeddings $f : \mathcal{N}_1 \to \mathcal{M}'$ and $g : \mathcal{N}_2 \to \mathcal{M}'$ such that $f(\mathcal{N}_1)$, $g(\mathcal{N}_2) \prec_{\mathcal{K}} \mathcal{M}'$. and $f \restriction \mathcal{M} = g \restriction \mathcal{M}$.

Theorem 4.2.18. *$\mathcal{K}_{(H,\Gamma_Q)}$ has the AP.*

Proof. Let (H_1, Γ_{Q_1}), (H_2, Γ_{Q_2}) and $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be such that $(H_1, \Gamma_{Q_1}) \prec (H_2, \Gamma_{Q_2})$ and $(H_1, \Gamma_{Q_1}) \prec (H_3, \Gamma_{Q_3})$. By Corollary 4.2.7, there are sets $G_1 \subseteq H_1$, $G_{23} \subseteq H_2 \cap H_3$, $G_2 \subseteq H_2$ and $G_3 \subseteq H_3$ such that:

- $G_1 \subseteq G_{23}$
- $H_1 = H_d \oplus \bigoplus_{v\in G_1}(H_1)_v$
- $H_2 \cap H_3 = H_d \oplus \bigoplus_{v\in G_{23}}(H_2)_v$
- $H_2 = H_d \oplus \bigoplus_{v\in G_{23}}(H_2)_v \oplus \bigoplus_{v\in G_2}(H_2)_v$
- $H_3 = H_d \oplus \bigoplus_{v\in G_{23}}(H_3)_v \oplus \bigoplus_{v\in G_3}(H_3)_v$

Let

$$H_4 := H_d \oplus \bigoplus_{v\in G_{23}}(H_2)_v \oplus \bigoplus_{v\in G_2}(H_2)_v \oplus \bigoplus_{v\in G_3}(H_3)_v$$

and

$$Q_4 := (Q_1 \restriction H_d) \oplus \Big(\bigoplus_{v\in G_{23}}(Q_2 \restriction (H_2)_v)\Big) \oplus \Big(\bigoplus_{v\in G_2}(Q_2 \restriction (H_2)_v)\Big) \oplus \Big(\bigoplus_{v\in G_3}(Q_3 \restriction (H_3)_v)\Big)$$

We have that $\sigma(Q_1 \restriction H_d) = \sigma_d(Q)$, $(H_4) \simeq H_d$, $\sigma\big(\bigoplus_{v\in G_{23}}(Q_2 \restriction (H_2)_v)\big) = \sigma\big(\bigoplus_{v\in G_2}(Q_2 \restriction (H_2)_v)\big) = \sigma\big(\bigoplus_{v\in G_3}(Q_3 \restriction (H_3)_v)\big) = \sigma_e(Q)$. Then $(H_4, \Gamma_{Q_4}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $Id_{H_d} \oplus \bigoplus_{v\in G_1} Id_{(H_1)_v} \oplus \bigoplus_{v\in G_2} Id_{(H_2)_v}$, $id_{H_d} \oplus \bigoplus_{v\in G_1} Id_{(H_1)_v} \oplus \bigoplus_{v\in G_3} Id_{(H_3)_v}$ are respective $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings from (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) to (H_4, Γ_{Q_4}). □

Remark 4.2.19. For (H_1, Γ_{Q_1}), (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) as in Theorem 4.2.18, we denote by

$$(H_2, \Gamma_{Q_2}) \bigvee_{(H_1,\Gamma_{Q_1})} (H_3, \Gamma_{Q_3}) := (H_2 \vee_{H_2} H_3, \Gamma_{Q_2 \vee_{Q_1} Q_3})$$

the *amalgamation* of (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) over (H_1, Γ_{Q_1}) as described in Theorem 4.2.18.

Definition 4.2.20. For $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$, $A \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$ and $(a_i)_{i<\alpha} \subseteq \mathcal{M}_1$, $(b_i)_{i<\alpha} \subseteq \mathcal{M}_2$, we say that $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ have the same *Galois type over* A in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, ($gatp_{\mathcal{M}_1}((a_i)_{i<\alpha}/A) = gatp_{\mathcal{M}_2}((b_i)_{i<\alpha}/A)$), if there are $\mathcal{N} \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$ such that $f(a_i) = g(b_i)$ for every $i < \alpha$ and $f \restriction A \equiv g \restriction A \equiv Id_A$, where Id_A is the identity on A.

Theorem 4.2.21. *Let* $v \in (H_1, \Gamma_{Q_1})$, $w \in (H_2, \Gamma_{Q_2})$ *and* $G \subseteq H_1 \cap H_2$ *such that* $(H_G, \Gamma_{Q_G}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $(H_G, \Gamma_{Q_G}) \prec (H_1, \Gamma_{Q_1})$, $(H_G, \Gamma_{Q_G}) \prec (H_2, \Gamma_{Q_2})$. *Then* $gatp_{(H_1,\Gamma_{Q_1})}(v/G) = gatp_{(H_2,\Gamma_{Q_2})}(w/G)$ *if and only if*

$$P_G v = P_G w$$

and

$$\mu_{P_{G^\perp} v} = \mu_{P_{G^\perp} w}.$$

Proof. **⇒)** Suppose $gatp_{(H_1,\Gamma_{Q_1})}(v/G) = gatp_{(H_2,\Gamma_{Q_2})}(w/G)$ and let $v' := P_{G^\perp}v$ and $w' := P_{G^\perp}w$. Then, by Definition 4.2.20, there exists $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings $U_1 : (H_1, \Gamma_{Q_1}) \to (H_3, \Gamma_{Q_3})$ and $U_2 : (H_2, \Gamma_{Q_2}) \to (H_3, \Gamma_{Q_3})$ such that $U_1 v = U_2 w$ and $U_1 \restriction G \equiv U_2 \restriction G \equiv Id_G$, where Id_G is the identity on G. Since $v = P_G v + P_{G^\perp} v$, $w = P_G w + P_{G^\perp} w$ and $U_1 \restriction G \equiv U_2 \restriction G \equiv Id_G$, we have that $U_1 P_G v = P_G v$ and $U_2 P_G w = P_G w$. This implies that $P_G v = P_G w$. On the other hand, since U_1 and U_2 are embeddings, $\mu_{v'} = \mu_{U_1 v'} = \mu_{U_2 w'} = \mu_{w'}$.

⇐) Let $v' := P_{G^\perp}v$ and $w' := P_{G^\perp}w$. Suppose $\mu_{v'} = \mu_{w'}$, then $\mu_{v'_e} = \mu_{w'_e}$ and $L^2(\mathbb{R}, \mu_{v'_e}) = L^2(\mathbb{R}, \mu_{w'_e})$. Let $\mu := \mu_{v'_e} = \mu_{w'_e}$. Also, let

$$\hat{H} := (H_1 \vee_{H_G} H_2) \oplus L^2(\mathbb{R}, \mu)$$

and let

$$\hat{Q} := (Q_1 \vee_{Q_G} Q_2) \oplus M_{f_\mu}$$

be as in the Spectral Theorem-Multiplication form. Let $U_1 : (H_1, \Gamma_{Q_1}) \to (\hat{H}, \hat{Q})$ be the $\mathcal{K}_{(H,\Gamma_Q)}$-embedding acting on $H_{v'}^\perp$ into $H_{v'}^\perp \vee H_{w'}^\perp$ as in the AP, and acting on $H_{v'}$ as in Fact .4.37. Define $U_2 : (H_2, \Gamma_{Q_2}) \to (\hat{H}, \hat{Q})$ in the same way. Then, we have completed the conditions to show that $gatp_{(H_1,\Gamma_{Q_1})}(v/G) = gatp_{(H_2,\Gamma_{Q_2})}(w/G)$.

□

Definition 4.2.22. A MAEC $\mathcal{K}$ is said to be *homogeneous* if whenever $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ and $(a_i)_{i<\alpha} \subseteq \mathcal{M}$, $(b_i)_{i<\alpha} \subseteq \mathcal{N}$ are such that for all $n < \omega$ and $i_0, \ldots, i_{n-1} < \alpha$

$$gatp_{\mathcal{M}}(a_{i_0}, \ldots, a_{i_{n-1}}/\emptyset) = gatp_{\mathcal{N}}(b_{i_0}, \ldots, b_{i_{n-1}}/\emptyset)$$

then we have that

$$gatp_{\mathcal{M}}((a_i)_{i<\alpha}/\emptyset) = gatp_{\mathcal{N}}((b_i)_{i<\alpha}/\emptyset),$$

Theorem 4.2.23. *$\mathcal{K}_{(H,\Gamma_Q)}$ is a homogeneous MAEC.*

Proof. Let $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $(v_i)_{i<\alpha} \subseteq H_1$, $(w_i)_{i<\alpha} \subseteq H_2$ be such that for all $n < \omega$ and $i_0, \ldots, i_{n-1} < \alpha$

$$gatp_{(H_1,\Gamma_{Q_1})}(v_{i_0}, \ldots, v_{i_{n-1}}/\emptyset) = gatp_{(H_2,\Gamma_{Q_2})}(w_{i_0}, \ldots, w_{i_{n-1}}/\emptyset)$$

We can use Graham-Schmidt-like process to get orthonormal sequences. So, without loss of generality, we can assume that for all $i < \alpha$ $v_i \in (H_1)_e$, $w_i \in (H_2)_e$ and for every $i \neq j < \alpha$, $v_i \perp v_j$ and $w_i \perp w_j$. For $i < \alpha$, let $\mu_i := \mu_{v_i} = \mu_{w_i}$, which agree by Theorem 4.2.21, since for all $i < \alpha$ $gatp_{(H_1,\Gamma_{Q_1})}(v_i/\emptyset) = gatp_{(H_2,\Gamma_{Q_2})}(w_i/\emptyset)$. Also, let

$$\hat{H} := (H_1 \vee_\emptyset H_2) \oplus \bigoplus_{i<\alpha} L^2(\mathbb{R}, \mu_i)$$

and

$$\hat{Q} := (Q_1 \vee_\emptyset Q_2) \oplus \bigoplus_{i<\alpha} M_{f_{\mu_i}}$$

be as in the Spectral Theorem-Multiplication form. Let $U_1 : (H_1, \Gamma_{Q_1}) \to (\hat{H}, \Gamma_{\hat{Q}})$ be the $\mathcal{K}_{(H,\Gamma_Q)}$-embedding acting on $H^\perp_{(v_i)_{i<\alpha}}$ into $H^\perp_{(v_i)_{i<\alpha}} \vee H^\perp_{(w_i)_{i<\alpha}}$ as in the AP, and acting on $H_{(v_i)_{i<\alpha}}$ as in Fact .4.37. Define $U_2 : (H_2, \Gamma_{Q_2}) \to (\hat{H}, \hat{Q})$ in the same way. Then we have completed the conditions to show that $gatp_{(H_1,\Gamma_{Q_1})}((v_i)_{i<\alpha}/\emptyset) = gatp_{(H_2,\Gamma_{Q_2})}((w_i)_{i<\alpha}/\emptyset)$. ☐

Theorem 4.2.24 (Theorem 1.13 in [18])**.** *Let $(\mathcal{K}, \prec_\mathcal{K})$ a MAEC on a similarity type Λ satisfying JEP, AP and homogeneity. Let $\kappa > |\Lambda| + LS(\mathcal{K})$, then there is $\mathfrak{M} \in \mathcal{K}$ such that*

(κ-universality) *$\mathfrak{M}$ is κ-universal, that is for all $\mathcal{M} \in \mathcal{K}$ such that $|\mathcal{M}| < \kappa$, there is a $\mathcal{K}$ embedding $f : \mathcal{M} \to \mathfrak{M}$.*

(κ-homogeneity) *$\mathfrak{M}$ is κ-homogeneous, that is if $(a_i)_{i<\alpha}$, $(b_i)_{i<\alpha} \subseteq \mathfrak{M}$ are such that for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$*

$$gatp_\mathfrak{M}(a_{i_0}, \dots, a_{i_{n-1}}/\emptyset) = gatp_\mathfrak{M}(b_{i_0}, \dots, b_{i_{n-1}}/\emptyset)$$

then there is an automorphism f of $\mathfrak{M}$ such that $f(a_i) = b_i$ for all $i < \alpha$.

Definition 4.2.25. If in the previous theorem, κ is a cardinal greater than the density of any structure in $\mathcal{K}$ that we want to study, the structure $\mathfrak{M}$ is called a weak *Monster Model.*

Remark 4.2.26. Let κ be as above, and let $\mathbb{M}(\mathbb{R})$ the set of all regular Borel measures on $\mathbb{R}$ whoose support is disjoint from $\sigma_p(Q)$. Then the structure $(\tilde{H}_\kappa, \Gamma_{\tilde{Q}_\kappa})$ where

$$\tilde{H} = H_d \oplus \bigoplus_\kappa \big(\bigoplus_{\mu \in \mathbb{M}} L^2(\mathbb{R}, \mu)\big)$$

and

$$\tilde{Q} = (Q \upharpoonright H_d) \oplus \bigoplus_\kappa \big(\bigoplus_{\mu \in \mathbb{M}} M_{f_\mu}\big)$$

works as a weak monster model for $\mathcal{K}_{(H,\Gamma_Q)}$. This can be easily proven from the proofs of JEP, AP and homogeneity of $K_{(H,\Gamma_Q)}$.

Definition 4.2.27. Let $\mathcal{K}$ be a MAEC that satisfies the JEP, AP and homogeneity. Let $\mathfrak{M}$ be a monster model for $\mathcal{K}$. Then $\mathcal{K}$ is said to have the *continuity of types property* if whenever $A \subseteq \mathfrak{M}$ and $(b_i)_{i<\omega}$ is a convergent sequence with limit $b = \lim_{n\to\infty} b_i$ such that $gatp(b_i/A) = gatp(b_j/A)$ for all i, $j < \omega$, then $gatp(b/A) = gatp(b_i/A)$ for all $i < \omega$.

Theorem 4.2.28. *$\mathcal{K}_{(H,\Gamma_Q)}$ has the continuity of types property.*

Proof. Let $G \subseteq \tilde{H}$ be small (i.e. $Dens(G) < Dens(H)$) and $(v_i)_{i<\omega} \subseteq \tilde{H}$ a sequence such that $\lim_{i\to\infty} v_i = v$ and $gatp(v_i/G) = gatp(v_j/G)$ for all $i,\ j < \omega$. Then by Theorem 4.2.21, $P_G v_i = P_G v_j$ and $gatp(P_{G^\perp} v_i/\emptyset) = gatp(P_{G^\perp} v_j/\emptyset)$ for all $i,\ j < \omega$. If $\lim_{i\to\infty} v_i = v$, it is clear that $P_G v_i = P_G v$ for all $i < \omega$. So it is enough to prove the theorem for the case $G = \emptyset$. Suppose $\lim_{i\to\infty} v_i = v$ and $gatp(v_i/\emptyset) = gatp(v_j/\emptyset)$ for all $i,\ j < \omega$. By Theorem 4.2.21, this means that $\mu_i = \mu_j$ for all $i,\ j < \omega$. Let $\mu := \mu_i$ and $E \subseteq \mathbb{R}$ be a Borel set. Then $\langle \chi_E(Q)v \mid v\rangle = \langle \chi_E(Q)(\lim_{i\to\infty} v_i \mid \lim_{i\to\infty} v_i\rangle = \lim_{i\to\infty}\langle \chi_E(Q)v_i \mid v_i\rangle = \lim_{i\to\infty} \mu_i(E) = \lim_{i\to\infty} \mu(E) = \mu(E)$. Again by Theorem 4.2.21, $gatp(v_i/\emptyset) = gatp(v/\emptyset)$ for all $i < \omega$. □

Remark 4.2.29. In the equalities used in the proof of previous theorem, we can exchange the limit with $\chi(Q)$ because $\chi(Q)$ is a bounded (and therefore continuous) operator.

4.3 Definable and algebraic closures

In this section we give a characterization of definable and algebraic closures.

Definition 4.3.1. Let $\mathcal{K}$ be a MAEC with JEP and AP. Let $\mathfrak{M}$ be the monster model in $\mathcal{K}$ and let $A \subseteq \mathfrak{M}$ be a small subset. Then,

1. The *definable closure* of A is the set

   $dcl(A) := \{m \in \mathfrak{M} \mid Fm = m$ for all F automorphism of $\mathfrak{M}$ that fixes A pointwise$\}$

2. The *algebraic closure* of A is the set

   $acl(A) := \{m \in \mathfrak{M} \mid$ the orbit under $Aut(\mathfrak{M}/A)$ is compact$\}$

Remark 4.3.2. Recall that $Aut(\mathfrak{M}/A)$ is the group of automorphisms of $\mathfrak{M}$ that fix A pointwise.

Theorem 4.3.3. *Let $G \subseteq \tilde{H}$. Then $dcl(G) = \tilde{H}_G$.*

Proof. $dcl(G) \subseteq \tilde{H}_G$ Let $v \notin \tilde{H}_G$. Then $P_{G^\perp} v \neq 0$. Let $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be a small structure containing v. Let $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be a structure containig $H' \oplus L^2(\mathbb{R}, \mu_{P_{G^\perp} v_e})$. Let $w := P_G v + (1)_{\mu_{P_{G^\perp} v_e}} \in H''$. Then $gatp(v/G) = gatp(w/G)$, but $v \neq w$. Therefore $v \notin dcl(G)$.

$\tilde{H}_G \subseteq dcl(G)$ Let $v \in G$, let h be a bounded Borel function on $\mathbb{R}$, let $U \in Aut(\tilde{H}, \tilde{Q}/G)$ and let $(H', \Gamma_{Q'})$ a small structure containg G. Then, by Lemma 4.2.8, $Uh(Q')v = h(Q')Uv = h(Q')v$, and $v \in dcl(G)$.

□

Lemma 4.3.4. *Let $v \in \tilde{H}$. If v is an eigenvector corresponding to some $\lambda \in \sigma_d(Q)$ then v is algebraic over $\emptyset$.*

Proof. $\lambda \in \sigma_d(Q)$ if and only if λ is isolated in $\sigma(Q)$ with finite dimensional eigenspace $\tilde{H}_\lambda$. So any automorphism can only send $\tilde{H}_\lambda$ onto $\tilde{H}_\lambda$ and the orbit of v under such automorphism can only be compact. □

Lemma 4.3.5. *Let $v \in \tilde{H}$ be such that $v = \sum v_k$ where each v_k is an eigenvector for some $\lambda_k \in \sigma_d(Q)$. Then v is algebraic over $\emptyset$.*

Proof. Given that $\|v_k\| \to 0$ when $k \to \infty$, the orbit of v under all the automorphisms is a Hilbert cube which is compact. □

Theorem 4.3.6. $acl(\emptyset) = H_d$

Proof. $acl(\emptyset) \subseteq H_d$ is a consecuence of Lemma 4.3.5. For the converse, suppose $v \in \tilde{H}$ such that $v_e \neq 0$. Let η be an uncountable small cardinal and let $F := \bigoplus_\eta L^2(\mathbb{R}, \mu_{v_e})$. Any structure in $\mathcal{K}_{(H,\Gamma_Q)}$ containing G will have η different realizations of $gatp(v/\emptyset)$. Therefore $v \notin acl(\emptyset)$. □

Theorem 4.3.7. *Let $G \subseteq \tilde{H}$. Then $acl(G)$ is closed Hilbert subspace generated by the union of $dcl(G)$ with $acl(\emptyset)$.*

Proof. Let E be the space $acl(\emptyset)+dcl(G)$. We have that $acl(\emptyset) \subseteq acl(G)$ and $dcl(G) \subseteq acl(G)$ so $E \subseteq acl(G)$. If $v \notin E$, then $P_E^\perp v \neq 0$. Let η be an uncountable small cardinal and let $F := \bigoplus_\eta L^2(\mathbb{R}, \mu_{(P_E^\perp v)_e})$. Any structure in $\mathcal{K}_{(H,\Gamma_Q)}$ containing G will have η different realizations of $gatp(v/G)$. Therefore, $v \notin acl(A)$. □

4.4 Perturbations

In this section, we define a system of perturbations for $\mathcal{K}_{(H,\Gamma_Q)}$ and show that $\mathcal{K}_{(H,\Gamma_Q)}$ is separably categorical up to this system of perturbations.

Definition 4.4.1. Let $(\mathcal{K}, \prec_\mathcal{K})$ be a MAEC. A class $(\mathbb{F}_\epsilon)_{\epsilon \geq 0}$ collections of bijective mappings between members of $\mathcal{K}$ is said to be a *system of perturbations* for $(\mathcal{K}, \prec_\mathcal{K})$ if

1. $\mathbb{F}_\delta \subseteq \mathbb{F}_\epsilon$ if $\delta < \epsilon$, $\mathbb{F}_0 = \bigcap_{\epsilon>0} \mathbb{F}_\epsilon$ and $\mathbb{F}_0$ is exactly the collection of real isomorphisms of structures in $\mathcal{K}$.
2. If $f : \mathcal{M} \to \mathcal{N}$ is in $\mathbb{F}_\epsilon$, then f is a e^ϵ-bi lipschitz mapping with respect to the metric i.e. $e^{-\epsilon} d(x, y) \leq d(f(x), f(y)) \leq e^\epsilon d(x, y)$ for all $x, y \in M$.
3. If $f \in \mathbb{F}_\epsilon$ then $f^{-1} \in \mathbb{F}_\epsilon$.
4. If $f \in \mathbb{F}_\epsilon$, $g \in \mathbb{F}_\delta$, and $dom(g) = rng(f)$ then $g \circ f \in \mathbb{F}_{\epsilon+\delta}$.
5. If $(f_i)_{i<\alpha}$ is an increasing chain of ϵ-isomorphisms, i.e. $f_i \in \mathbb{F}_\epsilon$, $f_i : \mathcal{M}_i \to \mathcal{N}_i$, $\mathcal{M}_i \prec_\mathcal{K} \mathcal{M}_{i+1}$, $\mathcal{N}_i \prec_\mathcal{K} \mathcal{N}_{i+1}$ and $f_i \subseteq f_{i+1}$ for every $i < \alpha$, then there is an ϵ-isomorphism $f : \overline{\bigcup_{i<\alpha} \mathcal{M}_i} \to \overline{\bigcup_{i<\alpha} \mathcal{N}_i}$ such that $f \restriction \mathcal{M}_i = f_i$ for all $i < \omega$.

If $(\mathbb{F}_\epsilon)_{\epsilon\geq 0}$ is a system of perturbations for $(\mathcal{K}, \prec_\mathcal{K})$, then $(\mathcal{K}, \prec_\mathcal{K}, (\mathbb{F}_\epsilon)_{\epsilon\geq 0})$ is called a *MAEC with perturbations.*

Definition 4.4.2. Let $\epsilon > 0$. An *ϵ-perturbation* in $\mathcal{K}_{(H,\Gamma_Q)}$ is an unitary operator $U : H_1 \to H_2$ such that there are closed unbounded selfadjoint operators Q_1 and Q_2 defined on H_1 and H_2 respectively, such that

1. (H_1, Γ_{Q_1}), $(H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$
2. $UD(Q_1) = D(Q_1)$
3. The operator $Q_1 - U^{-1}Q_2U$ can be extended to a bounded operator on H_1 with norm less or equal to ϵ.
4. The operator $Q_2 - UQ_1U^{-1}$ can be extended to a bounded operator on H_2 with norm less or equal to ϵ.

The class of all ϵ-perturbations in $\mathcal{K}_{(H,\Gamma_Q)}$ is denoted by $(\mathbb{F}_\epsilon^{(H,\Gamma_Q)})_{\epsilon\geq 0}$

Theorem 4.4.3. $(\mathcal{K}_{(H,\Gamma_Q)}, \prec_{\mathcal{K}_{(H,\Gamma_Q)}}, (\mathbb{F}_\epsilon^{(H,\Gamma_Q)})_{\epsilon\geq 0})$ *is a MAEC with perturbations.*

Proof. Items (1), (2) and (3) are clear. (4) Comes from triangle inequality. For (5), recall from the Tarsky chain condition in Theorem 4.2.12 that $\overline{\bigcup_{i<\kappa}}(H_i, \Gamma_{Q_i}) = H_0 \bigoplus_{i<\kappa}(H_i', \Gamma_{Q_i'})$. This with the fact that a direct sum of κ bounded operators with norm less than ϵ is still a bounded operator with norm less than ϵ. □

Definition 4.4.4. A MAEC with a system of perturbations $(\mathcal{K}, \prec_\mathcal{K}, (\mathbb{F}_\epsilon)_{\epsilon\geq 0})$ is said to be $\aleph_0$*-categorical up to the system of perturbations* $(\mathbb{F}_\epsilon)_{\epsilon\geq 0}$, if for all separable $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$ and for all $\epsilon > 0$, there is an $f_\epsilon \in \mathbb{F}_\epsilon$ such that $f_\epsilon : \mathcal{M}_1 \to \mathcal{M}_2$.

Theorem 4.4.5. $(\mathcal{K}_{(H,\Gamma_Q)}, \prec_{\mathcal{K}_{(H,\Gamma_Q)}}, (\mathbb{F}_\epsilon^{(H,\Gamma_Q)})_{\epsilon\geq 0})$ *is* $\aleph_0$*-categorical up to the system of perturbations.*

Proof. Let (H_1, Γ_{Q_1}), $(H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be separable. For each $\epsilon > 0$, we build a structure $(H_\epsilon, \Gamma_{Q_\epsilon})$, an ϵ-isomorphism $V_\epsilon : (H_1, \Gamma_{Q_1}) \to (H_\epsilon, \Gamma_{Q_\epsilon})$ and an ϵ-isomorphism $W_\epsilon : (H_2, \Gamma_{Q_2}) \to (H_\epsilon, \Gamma_{Q_\epsilon})$. So, $V_\epsilon W_\epsilon^*$ is a 2ϵ-isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). This shows that $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-categorical up to the system of perturbations.
Now, let us go to the construction of the V_ϵ's: Let $\epsilon > 0$ and let $(I_k)_{k\in\mathbb{Z}^+}$ be a family of disjoint connected subsets of $\mathbb{R}$ with diameter less than ϵ, which also cover $\sigma_e(Q)$. Let $(\lambda_k)_{k\in\mathbb{Z}^+} \subseteq \sigma(Q)$ be a set of inner points in each of the I_k's. For each $k \in \mathbb{Z}^+$, let $(H_k', \Gamma_{Q_k'})$ be an $\aleph_0$-dimensional structure such that Q_k' acts on H_k' as λ_k times the identity. Given I_k, both $\chi_{I_k}(Q_1)H_1$ and H_k' are separable and infinite dimensional. Therefore, there is an isomorphism:

$$V_\epsilon^k : (\chi_{I_k}(Q_1)H_1, \Gamma_{Q_1\restriction\chi_{I_k}(Q_1)H_1}) \to (H_k', \Gamma_{Q_k'})$$

Now, let $\lambda_{d_1}, \dots, \lambda_{d_{n_\epsilon}}$ be the (finite) set of discrete spectral values (isolated finite dimensional eigenvalues) not covered by $(I_k)_{k\in\mathbb{Z}^+}$. Let H_{d_i} be the eigenspace of λ_{d_i} and let n_{d_i} be the dimension of H_{d_i}. Let Q_i be the restriction of Q_1 to H_{d_i}.
Let

$$(H_\epsilon, \Gamma_{Q_\epsilon}) := \bigoplus_{i=1}^{n_\epsilon}(H_{d_i}, \Gamma_{Q_{d_i}}) \oplus \bigoplus_{k\in\mathbb{Z}^+}(H'_k, \Gamma_{Q'_k})$$

and let

$$V_\epsilon :=:= \bigoplus_{i=1}^{n_\epsilon} Id_{H_{d_i}} \oplus \bigoplus_{k\in\mathbb{Z}^+} V_\epsilon^k.$$

Given that $|x - \sum_{i=1}^{n_\epsilon} \lambda_{d_i}\chi_{\{\lambda_{d_i}\}} - \sum_{k\in\mathbb{Z}^+}\lambda_k\chi_{I_k}| < \epsilon$, we get that $\|Q_1 - V_\epsilon^* Q_\epsilon V_\epsilon\| < \epsilon$. So, we have completed the proof. □

Remark 4.4.6. The previous theorem implies that any two separable structures (H_1, Γ_{Q_1}), $(H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ are approximately unitarily equivalent.

4.5 CFO elementary equivalence and continuous $\mathcal{L}_{\omega_1,\omega}$ axiomatization

In this section we deal with continuous first order elementary equivalence for the structures of the type (H, Γ_Q). We give a characterization of first order elementary equivalence and give a continuous $L_{\omega_1,\omega}$ axiomatization of the class $\mathcal{K}_{(H,\Gamma_Q)}$. As a by product of this, we get an alternate proof of an important cosequence of Weyl-von Neumann-Berg that states that two operators are approximately unitarily equivalent if and only if their essential and discrete spectra coincide and the dimensions of the eigenspaces of their eigenvalues are the same. This fact is proved by using $\aleph_0$-categoricity up to the system of perturbations proved in Section 4.4.

Lemma 4.5.1. *For every bounded linear operator $S \in B(H)$, definable in (H, Γ_Q), and for all v and $w \in H$, we have that $\|Sv - w\| \leq (2 + \|S\|)\Gamma_S(v, w)$ where $\Gamma_S(v, w)$ denotes the distance to the graph of S.*

Proof. Let G_S be the Hilbert subspace of $H \times H$ given by $G_S := \{(v, Sv) \mid v \in H\}$ and let P_S be the projection $H \times H$ over G_S. If $(v', Sv') := P_S(v, w)$, then $\Gamma_S(v, w) = d[(v', Sv'), (v, w)]$. So, $\|Sv - w\| \leq \Gamma_S(v, w) + d[(v', Sv'), (v, Sv)] \leq \Gamma_S(v, w) + d(v', v) + d(Sv', Sv) \leq \Gamma_S(v, w) + \Gamma_S(v, w) + \|S\|d(v', v) \leq 2\Gamma_S(v, w) + \|S\|\Gamma_S(v, w) = (2 + \|S\|)\Gamma_S(v, w)$. □

Lemma 4.5.2. *For every bounded linear operator $S \in B(H)$, definable in (H, Γ_Q), the following condition holds in (H, Γ_Q):*

$$\sup_v \sup_{w_1,w_2} \left(\|\tfrac{w_1-w_2}{2}\| \dot{-} (2 + \|S\|)\tfrac{\Gamma_S(v,w_1)+\Gamma_S(v,w_2)}{2}\right) = 0$$

Proof. Let $\bar{v}_1 := (v, w_1)$ and $\bar{v}_2 := (v, w_2)$, two pairs in $H \times H$. Then $\|w_1 - w_2\| \leq \|Sv - w_1\| + \|Sv - w_2\| \leq (2 + \|S\|)\Gamma_S(v, w_1) + (2 + \|S\|)\Gamma_S(v, w_2)$. □

Lemma 4.5.3. *For every closed linear operator S on H, definable in (H, Γ_Q), the following condition holds in (H, Γ_Q):*

$$\sup_{v_1,v_2,w_2,w_3} \Big(\Gamma_S^2(\tfrac{v_1+v_2}{2}, \tfrac{w_1+w_2}{2}) \dot{-} \big(\tfrac{\Gamma_S(v_1,w_1)+\Gamma_S(v_2,w_2)}{2}\big)^2\Big) = 0$$

Proof. Let $\bar{v}_1 := (v_1, w_1)$ and $\bar{v}_2 := (v_2, w_2)$, be two pairs in $H \times H$. Let $\bar{v}_1' := (v_1', w_1')$ and let $\bar{v}_2' := (v_2', w_2')$ be pairs in $H \times H$ such that $\Gamma_S(v_1', w_1') = \Gamma_S(v_2', w_2') = 0$. Then $d(\frac{\bar{v}_1+\bar{v}_2}{2}, \frac{\bar{v}_1'+\bar{v}_2'}{2}) \leq \frac{d(\bar{v}_1,\bar{v}_1')+d(\bar{v}_2,\bar{v}_2')}{2}$. Notice that, since v_1' and v_2' belong to the domain of S, so does $\frac{v_1'+v_2'}{2}$. So, $\big(d(\frac{\bar{v}_1+\bar{v}_2}{2}, \frac{\bar{v}_1'+\bar{v}_2'}{2})\big)^2 \leq \big(\frac{d(\bar{v}_1,\bar{v}_1')+d(\bar{v}_2,\bar{v}_2')}{2}\big)^2$.
Now, since S is closed, there exist $(\bar{v}_1^k := (v_1^k, w_1^k))_{k\in\mathbb{N}}$ and $(\bar{v}_2^k := (v_2^k, w_2^k))_{k\in\mathbb{N}}$, two sequences of pairs in $H \times H$ such that $\Gamma_S(v_1, w_1) = \lim_{k\to\infty} d[(v_1, w_1), (v_1^k, w_1^k)]$ and $\Gamma_S(v_2, w_2) = \lim_{k\to\infty} d[(v_2, w_2), (v_2^k, w_2^k)]$. Replacing in previous inequality and taking limits, we get the desired result. □

Next theorem is an adaptation of one already developed by Argoty and Ben Yaacov in [5]:

Theorem 4.5.4. *Let h be a bounded (complex) Borel function on $\mathbb{R}$. Then $\Gamma_{h(Q)}$ is definable in (H, Γ_Q) if and only if $h \in \mathcal{C}(\sigma(Q), \mathbb{C})$.*

Proof. In this proof $\prec$ will denote the usual first order elementary inclusion. Also, $(\hat{H}, \Gamma_{\hat{Q}})$ will denote a first order elementary extension of (H, Γ_Q) which is saturated and homogeneous.

$\Rightarrow$) Suppose h is a bounded Borel function on $\mathbb{R}$ which is not continuous on $\sigma(Q)$ and such that $\Gamma_{h(Q)}$ is definable in in (H, Γ_Q). Let $\lambda_0 \in \sigma(Q)$ be a point of discontinuity of h. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\sigma(Q)$ and $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$ such that $\lim_{\mathcal{U}} \lambda_k = \lambda_0$ and such that $\lim_{\mathcal{U}} h(\lambda_k)$ exists but $\lim_U h(\lambda_k) \neq h(\lambda_0)$. There exist models $(\mathcal{H}_k, \Gamma_{Q_k}) \prec (\hat{H}, \Gamma_{\hat{Q}})$ and $v_k \in \mathcal{H}_k$ for $k \in \mathbb{N}$ such that $\mathcal{H}_k \models \Gamma_Q(v_k, \lambda_k v_k) = 0$. Let $\mathcal{H} = \Pi_{\mathcal{U}}\mathcal{H}_k$ and let $v = (v_k)_{\mathcal{U}} \in \mathcal{H}$. Then $(v_k)_{\mathcal{U}}$ is an eigenvector in $\mathcal{H}$ for the eigenvalue λ_0, and we have:

$$\begin{aligned} h(\lambda_0)v = h(Q)(v) = h(Q)(v_k)_{\mathcal{U}} = (h(Q)v_k)_{\mathcal{U}} = \\ = (\lambda_k v_k)_{\mathcal{U}} = (\lim_U h(\lambda_k))(v_k)_{\mathcal{U}} = (\lim_U h(\lambda_k))v \end{aligned}$$

So $h(\lambda_0) = \lim_{\mathcal{U}} h(\lambda_k)$ which is a contradiction.

$\Leftarrow$) Suppose $h \in \mathcal{C}(\sigma(Q), \mathbb{C})$. Then by the Stone-Wierstrass theorem h can be uniformly approximated by a sequence of polynomials over $\sigma(Q)$. These polynomials are tranlated into polynomials in Q. Such polyinomials are definable, so $h(Q)$ is definable. □

Lemma 4.5.5. *If $\lambda \in \sigma(Q)$, λ is isolated if and only if $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Γ_Q).*

Proof. If $\lambda \in \sigma(Q)$, then $\chi_{\{\lambda\}}$ is continuous on $\sigma(Q)$ if and only if λ is isolated in $\sigma(Q)$. By Theorem 4.5.4, $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Γ_Q) if and only if $\chi_{\{\lambda\}}$ is continuous on $\sigma(Q)$. □

Lemma 4.5.6. *$\lambda \in \sigma_e(Q)$ if and only if for every $n \in \mathbb{N}$ and every bounded Borel function $h : \mathbb{R} \to \mathbb{C}$ such that h is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$, the condition:*

$$\inf_{v_1 v_2 \cdots v_n} \inf_{w_1, w_2 \cdots w_n} \max_{i,j=1,\cdots,n} \left(|\langle w_i \mid w_j\rangle - \delta_{ij}|, \|h(Q)v_i - w_i\|\right) = 0 \tag{4-1}$$

holds in (H, Q).

Proof. ⇒ Suppose $\lambda \in \sigma_e(Q)$ and let $h : \mathbb{R} \to \mathbb{C}$ be a bounded Borel function such that h is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$. Then there is an open set $V \subseteq \mathbb{R}$ such that $\lambda \in V$ and h does not have any zero in V. Even more, we can choose h such that there is an $M > 0$ such that $|h| > M$. Since $\lambda \in \sigma_e(Q)$, the space $\chi_V(Q)H$ is infinite dimensional and since h does not have any zero in V, there is a function h^{-1} which is continuous on V, $h^{-1}h \equiv 1$ (in the multiplicative sense) on V and h^{-1} can be extended contiuously on $\mathbb{R}$. By the Spectral Decomposition Theorem-Functional Calculus Form, $h(Q)h^{-1}(Q) \equiv Id_{\chi_V(Q)H}$ where $Id_{\chi_V(Q)H}$ is the identity operator on $\chi_V(Q)H$. This implies that $h(Q)$ is invertible in $\chi_V(Q)H$ and therefore the dimension of $h(Q)\chi_V(Q)H = h(Q)H$ is infinite.

On the other hand, by Theorem 4.5.4, the condition in the Equation 4-1 has sense in continuous first order logic, and corresponds to the first order sentence:

$$\exists v_1 v_2 \cdots v_n \exists w_1 w_2 \cdots w_n \left(\langle w_i | w_j\rangle = \delta_{ij} \wedge h(Q)v_i = w_i\right),$$

where δ_{ij} is Kronecker's delta. This condition states that $h(Q)H$ has dimension grater than n.

⇐ Suppose that for every $n \in \mathbb{N}$, and every bounded Borel function $h : \mathbb{R} \to \mathbb{C}$ such that f is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$, we have that Equation 4-1 holds. Let $\epsilon > 0$ and let h_n be a sequence of continuous functions on $\mathbb{R}$ that converge to $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}$. By the Spectral Decomposition Theorem-Functional Calculus Form, $h_n(Q) \to \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)$ in the norm. Since $h_n(Q)H$ is infinite dimensional for all $n \in \mathbb{N}$, $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)H$ is infinite dimensional. Since $\epsilon > 0$ is arbitrary, by Theorem .4.29, $\lambda \in \sigma_e(Q)$.

□

Lemma 4.5.7. *If λ is a complex number. Then $\lambda \in \sigma_d(Q)$ if and only if there exists $n \in \mathbb{N}$ such that the following condition exists in continuous first order logic and is true in (H, Γ_Q):*

$$\inf_{v_1 v_2 \cdots v_n} \sup_{w} \max\Big(|\langle v_i | v_j\rangle - \delta_{ij}|, \Gamma_Q(v_i, \lambda v_i), \qquad \Big\|\chi_{\{\lambda\}}(Q)w - \sum_{k=1}^{n} \langle w | v_i\rangle v_i\Big\|\Big) = 0 \tag{4-2}$$

where δ_{ij} is Kronecker's delta.

Proof. By Lemma 4.5.5, λ is isolated in $\sigma(Q)$ if and only if $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Q). Then, Condition 4-2 has sense in continuous first order logic if and only if λ is isolated in $\sigma(Q)$. On the other hand, Condition 4-2 is a continuous first order condition corresponding to

$$\exists v_1 v_2 \cdots v_n \forall w \Big((\bigwedge_{i,j} \langle v_i | v_j \rangle = \delta_{ij}) \wedge (\bigwedge_{i=1}^{n} Qv_i = \lambda v_i) \wedge \chi_{\{\lambda\}}(Q)w = \sum_{k=1}^{m} \langle w | v_i \rangle v_i \Big) = 0$$

In particular, the statement $\chi_{\{\lambda\}}(Q)w = \sum_{k=1}^{m} \langle w|v_i\rangle v_i$ means that the vectors $v_1, \cdots, v_n$ generate the eigenspace of λ. So the dimension of the eigenspace of λ is n. □

Lemma 4.5.8. *If λ is a complex number, then $\lambda \notin \sigma(Q)$ if and only if for some $c > 0$ and for some continuous function $f : [0, 1] \to [0, 1]$ such that $f(0) = 0$, the following condition is true in (H, Γ_Q):*

$$\sup_v \sup_w \Big((c\|v\| \dot{-} \|w\|) \dot{-} f(\Gamma_Q(v, \lambda v + w)) \Big) = 0 \tag{4-3}$$

Proof. $\Rightarrow$ Suppose $\lambda \notin \sigma(Q)$. By Fact .4.17 there exists $c > 0$ such that for every $v \in D(Q)$, $\|(Q - \lambda I)v\| \geq c\|v\|$. Given $r \in [0, 1]$, let $f(r) := \sup\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = r\}$. The function f is well defined, since the set $\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = r\}$ is bounded in $\mathbb{R}$ for all $r \in [0, 1]$, and is also continuous on $[0, 1]$. Now, $f(0) = sup\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = 0\}$; the condition $\Gamma_Q(v, \lambda v + w) = 0$ means that $v \in D(Q)$ and $Qv = \lambda v + w$, which means that $w = Qv - \lambda v$. So, $w = R_\lambda v$ and by Theorem .4.17, $\|w\| \geq c\|w\|$ thus $c\|v\| \dot{-} \|w\| = 0$. Therefore $f(0) = 0$.

$\Leftarrow$ Suppose now that Condition 4-3 holds for some $c > 0$ and $f : [0, 1] \to [0, 1]$ continuous such that $f(0) = 0$. Then if $v \in D(Q)$ and $w := (Q - \lambda I)v$, $\Gamma_Q(v, \lambda v + w) = 0$ and since $f(0) = 0$, $f(\Gamma_Q(v, \lambda v + w)) = 0$. By Condition 4-3, we have that $c\|v\| \dot{-} \|w\| = 0$ and therefore $c\|v\| \leq \|w\|$ what, by Theorem .4.17, means that $\lambda \in \rho(Q)$

□

Next theorem is a remark from C.W. Henson:

Theorem 4.5.9. *Let Q_1 and Q_2 be two closed (unbounded) self adoint operators on the separable Hilbert space H. Then the following statements are equivalent:*

1. *Q_1 and Q_2 are approximately unitarily equivalent.*
2. *The structures (H, Γ_{Q_1}) and (H, Γ_{Q_2}) are elementarily equivalent.*
3. *$Q_1 \sim_\sigma Q_1$.*

Proof. (1)→(2) Suppose that Q_1 and Q_2 are approximately unitarily equivalent. Then there exists a sequence of unitary operators U_n on H such that $\lim_{n\to\infty} U_n Q_1 U_n^* = Q_2$. Let $\mathbb{N}$ be an ultrafilter over $\mathbb{N}$ which contains the filter of cofinite subsets of $\mathbb{N}$. Let $(\hat{H}_1, \Gamma_{\hat{Q}_1}) = \Pi_{\mathbb{N}}(H, \Gamma_{U_n Q_1 U_n^*})$ and let $(\hat{H}_2, \Gamma_{\hat{Q}_2}) = \Pi_{\mathbb{N}}(H, \Gamma_{Q_2})$. It follows that $(\hat{H}_1, \Gamma_{\hat{Q}_1}) \simeq (\hat{H}_2, \Gamma_{\hat{Q}_2})$ and by Keisler-Shelah's theorem, $(H, \Gamma_{Q_1}) \equiv (H, \Gamma_{Q_2})$.

(2)→(3) Suppose $(H, \Gamma_{Q_1}) \equiv (H, \Gamma_{Q_2})$. Since the relation $Q_1 \sim_\sigma Q_2$ can be written down as sets of conditions in continuous first order logic (see Lemma 4.5.6, Lemma 4.5.7 and Lemma 4.5.8), we have that $Q_1 \sim_\sigma Q_2$.

(3)→(1) Suppose now that $Q_1 \sim_\sigma Q_2$. Then $(H, \Gamma_{Q_2}) \in \mathcal{K}_{(H_1, \Gamma_{Q_1})}$. By Theorem 4.4.5 and Remark 4.4.6, Q_1 and Q_2 are approximately unitarily equivalent.

□

Definition 4.5.10. Let $IHS_{\sigma(Q)}$ the theory of Hilbert spaces together with the following conditions in continuous $\mathcal{L}_{\omega_1\omega}$:

1. For every continuous bounded function $h : \sigma(Q) \to \mathbb{C}$:

$$\sup_v \sup_{w_1,w_2} \left(\|\tfrac{w_1-w_2}{2}\| \dot{-} (2 + \|h(Q)\|)^{\frac{\Gamma_{h(Q)}(v,w_1)+\Gamma_{h(Q)}(v,w_2)}{2}}\right) = 0$$

This expresses the fact that $h(Q)$ is a function (see Lemma 4.5.2).

2. •

$$\sup_{v_1,v_2,w_2,w_3} \left(\Gamma_Q^2\left(\tfrac{v_1+v_2}{2}, \tfrac{w_1+w_2}{2}\right) \dot{-} \left(\tfrac{\Gamma_Q(v_1,w_1)+\Gamma_Q(v_2,w_2)}{2}\right)^2\right) = 0$$

•

$$\sup_v \sup_w (|\Gamma_Q(v,w) - \Gamma_Q(iv, iw)| = 0$$

•

$$\sup_v \sup_w (|\Gamma_Q(v,w) - \Gamma_Q(-v, -w)| = 0$$

This conditions express the fact that Q is linear (see Lemma 4.5.3).

3.

$$\sup_w \max\{\inf_v \Gamma_Q(v, w + iv), \inf_v \Gamma_Q(v, w - iv)\} = 0$$

This expresses that Q is essentially self-adjoint (see Fact .4.16).

4.

$$\sup_v \inf_w \Gamma_Q(v,w) = 0$$

This expresses that $D(Q)$ is dense.

5. For every $\lambda \in \sigma_d(Q)$ and for $n_\lambda := \dim \chi_{\{\lambda\}}(Q)H$,

$$\inf_{v_1 v_2 \cdots v_{n_\lambda}} \sup_w \max\Big(|\langle v_i|v_j\rangle - \delta_{ij}|, \Gamma_Q(v_i, \lambda v_i), \big\|\chi_{\{\lambda\}}(Q)w - \sum_{k=1}^{n_\lambda}\langle w \mid v_i\rangle v_i\big\|\Big) = 0$$

6. For every $\lambda \in \sigma_e(Q)$, for every $n \in \mathbb{N}$ and for every bounded Borel function $h : \mathbb{R} \to \mathbb{C}$ such that h is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$ (see Lemma 4.5.6):

$$\inf_{v_1 v_2 \cdots v_n} \inf_{w_1, w_2 \cdots w_n} \max_{i,j=1,\cdots,n} (|\langle w_i \mid w_j\rangle - \delta_{ij}|, \|h(Q)v_i - w_i\|) = 0$$

7. For every $\lambda \in \rho(Q)$, let c_λ and $f_\lambda : [0,1] \to [0,1]$ be such that they satisfy the hypothesis as in Lemma 4.5.8,

$$\sup_v \sup_w \Big((c_\lambda\|v\| \dot{-} \|w\|) \dot{-} f(\Gamma_Q(v, \lambda v + w))\Big) = 0$$

Theorem 4.5.11. *The class $K_{(H,\Gamma_Q)}$ is exactly the class of all models of $IHS_{\sigma(Q)}$.*

Proof. All continuous first order axioms guarantee that all models of $IHS_{\sigma(Q)}$ are spectrally equivalent to (H, Γ_Q). Condition 3 says that, in each model $(H', \Gamma_{Q'})$ of $IHS_{\sigma(Q)}$, the operator Q' is essentially self-adjoint. The Condition $\Gamma_{Q'} = 0$ implies that the graph of Q' is closed, so Q is a closed operator. Condition 4 implies says that in each model of $IHS_{\sigma(Q)}$ the domain of the closed unbounded operator is dense in the Hilbert space. So, all the models of $IHS_{\sigma(Q)}$ belong to $K_{(H,\Gamma_Q)}$. By spectral theory, the converse is true, so both classes are the same. □

Remark 4.5.12. $IHS_{\sigma(Q)}$ is not a theory in continuous first order logic but in continuous $\mathcal{L}_{\omega_1\omega}$ logic.

Now, we provide an example of a class $K_{(H,\Gamma_Q)}$ that only has one model which clarifies why, in general $K_{(H,\Gamma_Q)}$ is not the same as the class of models of $Th(H, \Gamma_Q)$, the first order theory of (H, Γ_Q). This example is very similar to the quantum harmonic oscillator:

Example 4.5.13. Let $(H_\mathbb{N}, \Gamma_{Q_N})$ be a separable Hilbert space structure such that $\sigma(Q_N) = \sigma_d(Q) = \mathbb{N}$ and for every $n \in \mathbb{N}$ the eigenspace corresponding to n has dimension 1.

Claim. Let $\mathcal{U}$ a non principal ultrafilter over $\mathbb{N}$. Then $D(\Pi_\mathcal{U} Q_\mathbb{N})$ is not dense in $\Pi_\mathcal{U} H_\mathbb{N}$.

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $H_\mathbb{N}$ such that $\|v_n\| = 1$, $v_n \in D(Q_\mathbb{N})$ and $Qv_n = nv_n$ for every $n \in \mathbb{N}$. Then $(v_n)/\mathcal{U} \notin D(\Pi_\mathcal{U} Q_\mathbb{N})$. Let $(w_n)/\mathcal{U}$ be such that $\|(v_n)/\mathcal{U} - (w_n)/\mathcal{U}\| < \epsilon$, for some $\epsilon > 0$. Then $\lim_\mathcal{U} \|w_n - v_n\| < \epsilon$. That is, for some $B \in \mathcal{U}$ and for every $n \in B$, $\|w_n - v_n\| < \epsilon$. Suppose, in addition, that for every $n \in B$, $w_n \in D(Q_\mathbb{N})$. Let $w_n = \sum_{k\geq 0} w_n^k$, where $w_n^k \in D(Q_\mathbb{N})$ and $Qw_n^k = kw_n^k$ for $k \in \mathbb{N}$. Then, $0 \leq \sum_{k\in\mathbb{N},k\neq n} \|w_n^k\|^2 < \epsilon^2$, and $\|w_n^n - v_n\| < \epsilon$. If $\epsilon < \frac{\|v_n\|}{2}$, then $\|Q(w_n^n)\| = \|Q(w_n^n - v_n + v_n)\| = \|Q(v_n - (v_n - w_n^n))\| = \|Q(v_n) - Q(v_n - w_n^n)\| \geq |\|Q(v_n) - Q(v_n - w_n^n)\|| = \left|n\|v_n\| - \frac{n}{2}\|v_n\|\right| = \frac{n}{2}$. So, $\|Q(w_n)\| \geq \frac{n}{2}$. This means that $\lim_\mathcal{U} \|Q(w_n)\| = \infty$. So, $(w_n)/\mathcal{U} \notin D(\Pi_\mathcal{U} Q_\mathbb{N})$. This way, we have proven that for some $(v_n)/\mathcal{U} \in \Pi_\mathcal{U} H_\mathbb{N} \setminus D(\Pi_\mathcal{U} Q_\mathbb{N})$, there exists an $\epsilon > 0$ such that for every $(w_n)/\mathcal{U} \in \Pi_\mathcal{U} H_\mathbb{N}$ such that $\|(v_n)/\mathcal{U} - (w_n)/\mathcal{U}\| < \epsilon$, and $(w_n)/\mathcal{U} \notin D(\Pi_\mathcal{U} Q_\mathbb{N})$. This proves that $\Pi_\mathcal{U} Q_\mathbb{N}$ is not dense in $\Pi_\mathcal{U} H_\mathbb{N}$. □

Claim. $(\Pi_\mathcal{U} H_\mathbb{N}, \Gamma_{\Pi_\mathcal{U} Q_\mathbb{N}})$ does not belong to $K_{(H_\mathbb{N},\Gamma_{Q_\mathbb{N}})}$.

Proof. The previous claim shows that $D(\Pi_\mathcal{U} Q_\mathbb{N})$ is not dense in $\Pi_\mathcal{U} H_\mathbb{N}$, and then $(\Pi_\mathcal{U} H_\mathbb{N}, \Gamma_{\Pi_\mathcal{U} Q_\mathbb{N}})$ does not belong to $K_{(H_\mathbb{N},\Gamma_{Q_\mathbb{N}})}$. □

Claim. $K_{(H,\Gamma_Q)}$ is not, in general, first order axiomatizable.

Proof. Previous Theorem shows that $K_{(H_\mathbb{N},\Gamma_{Q_\mathbb{N}})}$ is not closed under ultrapowrers and, therefore, cannot be first order axiomatizable. □

4.6 Stability up to the systemn of perturbations and spectral independence

In this section we prove that $\mathcal{K}_{(H,\Gamma_Q)}$ has the so called *stability up to the system of perturbations*). This is proven in Theorem 4.6.3. Also, we define an independence relation in $\mathcal{K}_{(H,\Gamma_Q)}$, called *spectral independence*. Theorem 4.6.10 states that this relation has the same properties as non-forking for superstable first order theories, while Theorem 4.6.12 and Theorem 4.6.13 state that this relation characterize non-splitting.

Remark 4.6.1. Recall that if $G \subseteq \tilde{H}$ is small, $S(G)$ denotes the set of Galois types in one variable over G.

Definition 4.6.2. A MAEC with a system or perturbations $(\mathcal{K}, \prec_\mathcal{K}, (\mathbb{F}_\epsilon)_{\epsilon\geq 0})$ is said to be $\aleph_0$-*stable up to the system of perturbations* if for every separable structure $\mathcal{M} \in \mathcal{K}$ there is a separable $\mathcal{N} \succ_\mathcal{K} \mathcal{M}$, such that for every $\epsilon > 0$ and for every separable structure $\mathcal{N}' \succ_\mathcal{K} \mathcal{M}$, there is an ϵ-perturbation $f : \mathcal{N}' \to \mathcal{N}$ such that $f \restriction \mathcal{M} = Id_\mathcal{M}$.

Theorem 4.6.3. *$\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-stable up to the system of perturbations.*

Proof. Let $(H_0, \Gamma_{Q_0}) \in K_{(H,\Gamma_Q)}$ be separable. Let Λ be a countable dense subset of $\sigma_e(Q)$. Let $(H_1, \Gamma_{Q_1}) := (H_0, \Gamma_{Q_0}) \oplus \bigoplus_{\lambda \in \Lambda}(L^2(R, \delta_\lambda), M_\lambda)$. Let $(H_2, \Gamma_{Q_2}) \succ (H_0, \Gamma_{Q_0})$. Let (H_2', Γ_{Q_2}') be the orthogonal complement of (H_0, Γ_{Q_0}) in (H_2, Γ_{Q_2}). By Theorem 4.5.11 (H_1, Γ_{Q_1}) and (H_2', Γ_{Q_2}') are approximately uniformly equivalent and therefore there is an ϵ-perturbation relating (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). □

Remark 4.6.4. In previous proof, recall that M_λ is the multiplication by λ.

Definition 4.6.5. Let $v_1, \ldots, v_n \in \tilde{H}$ and let F, $G \subseteq \tilde{H}$. We say that $v_1, \ldots, v_n$ are *spectrally independent* from G over F if for all $i \leq n$ $P_{acl(F)}v_i = P_{acl(F \cup G)}v_i$ and denote it by $v_1, \ldots, v_n \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_F G$.

Remark 4.6.6. Let v, $w \in \tilde{H}$. Then v is independent from w over $\emptyset$ if and only if $\tilde{H}_{v_e} \perp \tilde{H}_{w_e}$ and denote it $v \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_\emptyset w$.

Remark 4.6.7. Let v, $w \in \tilde{H}$. Let $G \subseteq \tilde{H}$ be small. Then v is independent from w over G if and only if $\tilde{H}_{P^\perp_{acl(G)}(v)} \perp \tilde{H}_{P^\perp_{acl(G)}(w)}$ and denote it $v \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_G w$.

Remark 4.6.8. Let $\bar{v} \in H^n$ and E, $F \subseteq H$. Then $\bar{v} \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_E F$ if and only if for every $j = 1, \ldots, n$ $v_j \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_E F$ that is, for all $j = 1, \ldots, n$ $P_{acl(E)}(v_j) = P_{acl(E \cup F)}(v_j)$

Theorem 4.6.9. *Let $F \subseteq G \subseteq H$, $p \in S_n(F)$ $q \in S_n(G)$ and $\bar{v} = (v_1, \ldots, v_n)$, $\bar{w} = (v_1, \ldots, v_n) \in H^n$ be such that $p = gatp(\bar{v}/F)$ and $q = gatp(\bar{w}/G)$. Then q is an extension of p such that $\bar{w} \mathop{\smile\!\!\!\!\!\mid}\nolimits^*_F G$ if and only if the following conditions hold:*

1. *For every $j = 1, \ldots, n$, $P_{acl(F)}(v_j) = P_{acl(G)}(w_j)$*
2. *For every $j = 1, \ldots, n$, $\mu_{P^\perp_{acl(F)}v_j} = \mu_{P^\perp_{acl(G)}w_j}$*

Proof. Clear from Theorem 4.2.21 and Remark 4.6.7 □

Theorem 4.6.10. $\mathop{\smile\!\!\!\!\!\mid}\nolimits^*$ *satisfies:*

1. *Local character.*
2. *Finite character.*
3. *Transitivity of independence*
4. *Symmetry*
5. *Existence*
6. *Stationarity*

Proof. By Remark 4.6.8, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character Let $v \in H$ and $G \subseteq \tilde{H}$. Let $w = (P_{acl(G)}(v))_e$. Then there exist a sequence of $(l_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, a sequence $(f_1^k, \dots, f_{l_k}^k)_{k\in\mathbb{N}}$ of finite tuples of bounded Borel funtions of $\mathbb{R}$ and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k)_{k\in\mathbb{N}} \subseteq G$ such that if $w_k := \sum_{j=1}^{l_k} f_j^k(\tilde{Q})e_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$. Let $E_0 = \{e_j^k \mid j = 1, \dots, l_k \text{ and } k \in \mathbb{N}\}$. Then $v \downarrow^*_{E_0} E$ and $|E_0| = \aleph_0$.

Finite character We show that for $v \in H$, $E, F \subseteq \tilde{H}$, $v \downarrow^*_E F$ if and only if $v \downarrow^*_E F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\downarrow^*_E F$. Let $w = P_{acl(E\cup F)}(v) - P_{acl(E)}(v)$. Then $w \in \operatorname{acl}(E \cup F) \backslash \operatorname{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}^2$, a sequence $(g_1^k, \dots, g_{l_k+n_k}^k)_{k\in\mathbb{N}}$ of finite tuples of bounded Borel functions on $\mathbb{R}$, and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k, f_1^k, \dots, f_{n_k}^k)_{k\in\mathbb{N}}$ such that $(e_1^k, \dots, e_{l_k}^k) \subseteq E$, $(f_1^k, \dots, f_{n_k}^k)_{k\in\mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} g_j^k(\tilde{Q})e_i^k + \sum_{j=1}^{n_k} g_{l_k+j}^k(\tilde{Q})f_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$.

If $v \not\downarrow^*_E F$, then $w = P_{acl(E\cup F)}(v) - P_{acl(E)}(v) \neq 0$. For $\epsilon = \|w\| > 0$ there is k_ϵ such that if $k \geq k_\epsilon$ then $\|w - w_k\| < \epsilon$. Let $F_0 := \{f_1^1, \dots, f_{k_\epsilon}^{n_{k_\epsilon}}\}$ Then F_0 is a finite subset such that $v \not\downarrow^*_E F_0$.

Transitivity of independence Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \downarrow^*_E G$ then $P_{acl(E)}(v) = P_{acl(G)}(v)$. It is clear that $P_{acl(E)}(v) = P_{acl(F)}(v) = P_{acl(G)}(v)$ so $v \downarrow^*_E F$ and $v \downarrow^*_F G$. Conversely, if $v \downarrow^*_E F$ and $v \downarrow^*_F G$, we have that $P_{acl(E)}(v) = P_{acl(F)}(v)$ and $P_{acl(F)}(v) = P_{acl(G)}(v)$. Then $P_{acl(E)}(v) = P_{acl(G)}(v)$ and $v \downarrow^*_E G$.

Symmetry It is clear from Remark 4.6.7.

Invariance Let U be an automorphism of $(\tilde{H}, \Gamma_{\tilde{Q}})$. Let $\bar{v} = (v_1, \dots, v_n)$,$\bar{w} = (w_1, \dots, w_n) \in \tilde{H}^n$ and $G \subseteq \tilde{H}$ be such that $\bar{v} \downarrow^*_G \bar{w}$. By Remark 4.6.7, this means that for every $j,\ k = 1, \dots, n$ $\tilde{H}_{P^\perp_{acl(G)}(v_j)} \perp \tilde{H}_{P^\perp_{acl(G)}(w_k)}$. It follows that for every $j,\ k = 1, \dots, n$ $\tilde{H}_{P^\perp_{acl(UG)}(Uv_j)} \perp \tilde{H}_{P^\perp_{acl(UG)}(Uw_k)}$ and, again by Remark 4.6.7, $Uv \downarrow^*_{acl(UG)} Uw$.

Existence Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(F)$, there exists $q \in S_n(G)$ such that q is an $\downarrow^*$-independent extension of p.

Case $n = 1$ Let $v \in \tilde{H}$ be such that $p = gatp(v/F)$ and let $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be a structure containing v and G. Define

$$H'' := H' \oplus L^2(\mathbb{R}, \mu_{(P^\perp_{acl(F)}v)_e}),$$

$$Q'' := Q' \oplus M_{f_{(P^\perp_{acl(F)}v)_e}}$$

and

$$v' := P_{acl(F)}v + (1)_{\sim_{\mu_{(P^\perp_{acl(F)}v)_e}}}$$

Then $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $v' \in H''$ and, by Theorem 4.6.9, the type $gatp(v'/G)$ is a $\downarrow^*$-independent extension of $gatp(v/F)$.

Induction step Now, let $\bar{v} = (v_1, \ldots, v_n, v_{n+1}) \in \tilde{H}^{n+1}$. By induction hypothesis, there are $v'_1, \ldots, v'_n \in H$ such that $gatp(v'_1, \ldots, v'_n/G)$ is a $\downarrow^*$-independent extension of $gatp(v_1, \ldots, v_n/F)$. Let U be a monster model automorphism fixing F pointwise such that for every $j = 1, \ldots, n$, $U(v_j) = v'_j$. Let $v'_{n+1} \in \tilde{H}$ be such that $gatp(v'_{n+1}/Gv'_1 \cdots v'_n)$ is a $\downarrow^*$-independent extension of $gatp(U(v_{n+1})/Fv'_1, \cdots v'_n)$. Then, by transitivity, $gatp(v'_1, \ldots, v'_n, v'_{n+1}/G)$ is a $\downarrow^*$-independent extension of $gatp(v_1, \ldots, v_n, v_{n+1}/F)$.

Stationarity Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(F)$, if $q \in S_n(G)$ is a $\downarrow^*$-independent extension of p to G then $q = p'$, where p' is the $\downarrow^*$-independent extension of p to G built in the proof of existence.

Case $n = 1$ Let $v \in H$ be such that $p = gatp(v/F)$, and let $q \in S(G)$ and $w \in H$ be such that $w \models q$. Let v' be as in previous item. Then, by Theorem 4.6.9 we have that:

1. $P_{acl(F)}v = P_{acl(G)}v' = P_{acl(G)}w$
2. $\mu_{P^{\perp}_{acl(F)}v} = \mu_{P^{\perp}_{acl(G)}w} = \mu_{P^{\perp}_{acl(G)}v'}$

This means that $P_{acl(G)}v' = P_{acl(G)}w$, $\mu_{P^{\perp}_{acl(G)}w} = \mu_{P^{\perp}_{acl(G)}v'}$ and, therefore $q = tp(v'/G) = p'$.

Induction step Let $\bar{v} = (v_1, \ldots, v_n, v_{n+1})$, $\bar{v}' = (v'_1, \ldots, v'_n, v'_{n+1})$ and $\bar{w} = (w_1, \ldots, w_n) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $gatp(v'_1, \ldots, v'_n/G)$ and $gatp(w_1, \ldots, w_n/G)$ are $\downarrow^*$-independent extensions of $gatp(v_1, \ldots, v_n/F)$. By induction hypothesis, $gatp(v'_1, \ldots, v'_n/G) = gatp(w_1, \ldots, w_n/G)$. Let U be a monster model automorphism fixing F pointwise such that for every $j = 1, \ldots, n$, $U(v_j) = v'_j$ and let U' a monster model automorphism fixing G pointwise such that for every $j = 1, \ldots, n$, $U'(v_j) = w_j$. Again by transitivity,

$$gatp(U(w_{n+1})/Gv_1 \cdots v_n)$$

and

$$gatp(v'_{n+1}/Gv_1, \cdots v_n)$$

are $\downarrow^*$-independent extensions of $gatp(v_{n+1}/Gv_1, \cdots v_n)$.

By the case $n = 1$,

$$gatp(U^{-1}(v'_{n+1})/U^{-1}Gv_1 \cdots v_n) = gatp((U' \circ U)^{-1}(w_{n+1})/U^{-1}Gv_1, \cdots v_n)$$

and therefore

$$p' = gatp(v'_1, \ldots, v'_n v'_{n+1}/G) = gatp(w_1, \ldots, w_n, w_{n+1}/G) = q.$$

□

Definition 4.6.11. Let $\mathcal{K}$ be an homogeneous MAEC with monster model $\mathcal{M}$. Let $B \subseteq A \subseteq M$ and let $a \in M$. The type $gatp(a/A)$ is said to *split* over B if there are b, $c \in A$ such that

$gatp(b/B) = gatp(c/B)$

but

$gatp(b/Ba) \neq gatp(c/Ba)$

Theorem 4.6.12. *Let $v \in \tilde{H}$ and let $F \subseteq G \subseteq \tilde{H}$. If $gatp(v/G)$ splits over F then $v \not\downarrow^*_F G$.*

Proof. If $gatp(v/G)$ splits over F, then there are two vectors w_1 and $w_2 \in G$ such that $gatp(w_1/F) = gatp(w_2/F)$ but $gatp(w_1/Fv) \neq gatp(w_2/Fv)$. Then, either $gatp(P^\perp_{acl(Fv)}w_1/\emptyset) \neq gatp(P^\perp_{acl(Fv)}w_2/\emptyset)$ or $P_{acl(Fv)}w_1 \neq P_{acl(Fv)}w_2$. Let us consider each case:

Case $gatp(P^\perp_{acl(Fv)}w_1/\emptyset) \neq gatp(P^\perp_{acl(Fv)}w_2/\emptyset)$ Since

$$P^\perp_{acl(Fv)}w_1 = P^\perp_{acl(F)}w_1 - P_{P^\perp_{acl(F)}v_e}w_1$$

and

$$P^\perp_{acl(Fv)}w_2 = P^\perp_{acl(F)}w_2 - P_{P^\perp_{acl(F)}v_e}w_2,$$

this means that

$$gatp(P_{P^\perp_{acl(F)}v_e}w_1/\emptyset) \neq gatp(P_{P^\perp_{acl(F)}v_e}w_2/\emptyset)$$

So, either $P_{P^\perp_{acl(F)}v_e}w_1 \neq 0$ or $P_{P^\perp_{acl(F)}v_e}w_2 \neq 0$. Let us suppose without loss of generality that $P_{P^\perp_{acl(F)}v_e}w_1 \neq 0$. Then $P_{w_1}(P^\perp_{acl(F)}v_e) \neq 0$, which implies that $P_{acl(F)}v \neq P_{acl(Fw_1)}v$. That is, $v \not\downarrow^*_F w_1$ and by transitivity, $v \not\downarrow^*_F G$.

Case $P_{acl(Fv)}w_1 \neq P_{acl(Fv)}w_2$ Since

$$P_{acl(Fv)}w_1 = P_{acl(F)}w_1 + P_{P^\perp_{acl(F)}v_e}w_1$$

and

$$P_{acl(Fv)}w_2 = P_{acl(F)}w_2 + P_{P^\perp_{acl(F)}v_e}w_2,$$

this means that $P_{P^\perp_{acl(F)}v_e}w_1 \neq P_{P^\perp_{acl(F)}v_e}w_2$ and, therefore either $P_{P^\perp_{acl(F)}v_e}w_1 \neq 0$ or $P_{P^\perp_{acl(F)}v_e}w_2 \neq 0$. As in previous item, this implies that $v \not\downarrow^*_F G$.

□

Theorem 4.6.13. *Let $v \in \tilde{H}$ and $F \subseteq G \subseteq \tilde{H}$ such that $F = acl(F)$ and G is $|F|$-saturated. If $v \not\perp^*_F G$, then $gatp(v/G)$ splits over F.*

Proof. If $v \not\perp^*_F G$ then $w := P_G v - P_F v \neq 0$ and $w \perp F$. Since G is $|F|$-saturated, there is $w' \in G$ such that $gatp(w/F) = gatp(w'/F)$ and $w' \perp P_G v$. Since $\langle v \mid w\rangle \neq 0$, $P_v w \neq 0$, while $P_v w' = 0$. □

Definition 4.6.14. Let $\epsilon > 0$, $v \in \tilde{H}$ and let F, $G \subseteq \tilde{H}$. We say that v is *ϵ-spectrally independent* from G over F if $\|P_{acl(F\cup G)}v - P_{acl(F)}v\| \leq \epsilon$ and denote it $v \perp^\epsilon_F G$.

Theorem 4.6.15. *The relation $\perp^\epsilon$ satisfies the following properties:*

Local character *Let $v \in H$, $G \subseteq \tilde{H}$ and $\epsilon > 0$. Then there is a finite $G_0 \subseteq G$ such that $v \perp^\epsilon_{G_0} G$.*

Monotonicity of independence *Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$. If $v \perp^\epsilon_D G$ then $v \perp^\epsilon_E F$*

Proof. **Local character** Let $v \in H$, $G \subseteq \tilde{H}$ and $\epsilon > 0$. Let w, $(l_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, $(e^k_1, \dots, e^k_{l_k})_{k\in\mathbb{N}} \subseteq G$, $(f^k_1, \dots, f^k_{l_k})_{k\in\mathbb{N}}$ and w_k for $k \in \mathbb{N}$ be as in the proof of local character of $\perp^*$ in Theorem 4.6.10. Since $w_k \to w$ when $k \to \infty$, there is a $k_1 \in \mathbb{Z}$ such that $\|w_k - w\| < \epsilon$ for all $k \geq k_1$. Let $G_o := \{e^k_j \mid j = 1, \dots, l_k \text{ and } k \leq k_1\}$. Then, $v \perp^*_{G_0} G$.

Monotonicity of independence Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$ and $\epsilon > 0$. If $v \perp^\epsilon_D G$ then $\epsilon \geq \|P_{acl(D\cup G)}v - P_{acl(D)}v\| = \|P_{acl(G)}v - P_{acl(D)}v\| \geq \|P_{acl(f)}v - P_{acl(E)}v\|$. Therefore $v \perp^\epsilon_E F$.

□

Remark 4.6.16. Previous theorem shows that the class $\mathcal{K}_{(H,\Gamma_Q)}$ is superstable.

Definition 4.6.17. Let $\bar{v} = (v_1, \dots, v_n) \in H^n$ and $G \subseteq H$. A *canonical base* for the type $gatp(\bar{v}/G)$ is a set $F \subseteq H_G$ which is fixed pointwise by the parallelism class of Morley sequences in $gatp(\bar{v}/G)$ and such that $\bar{v} \perp^*_F G$.

Theorem 4.6.18. *Let $\bar{v} = (v_1, \dots, v_n) \in H^n$ and $G \subseteq H$. Then $Cb(gatp(\bar{v}/G)) := \{(P_G v_1, \dots, P_G v_n)\}$ is a canonical base for the type $gatp(\bar{v}/G)$.*

Proof. First of all, we consider the case of a 1-tuple. By Theorem 4.6.9 $gatp(v/G)$ does not fork over $Cb(gatp(v/G))$. Let $(v_k)_{k<\omega}$ a Morley sequence for $gatp(v/G)$. We have to show that $P_G v \in dcl((v_k)_{k<\omega})$. By Theorem 4.6.9, for every $k < \omega$ there is a vector w_k such that $v_k = P_G v + w_k$ and $w_k \perp acl(\{P_G v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_e$ and for all j, $k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v'_k := \frac{v_1+\dots+v_k}{n} = P_G v + \frac{w_1+\dots+w_k}{n}$. Then for every $k < \omega$, $v'_k \in dcl((v_k)_{k<\omega})$. Since $v'_k \to P_e v$ when $k \to \infty$, we have that $P_G v \in dcl((v_k)_{k<\omega})$.

For the case of a general n-tuple, by Remark 4.6.8, it is enough to repeat previous argument in every component of $\bar{v}$. □

4.7 Orthogonality and domination

In this section, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures.

Theorem 4.7.1. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\mu_{v_e} \perp \mu_{w_e}$.*

Proof. $p \perp^a q$ if and only if $\tilde{H}_{v_e'} \perp \tilde{H}_{w_e'}$ for all $v_e' \models p$ and $w_e' \models q$. By Lesbesgue decomposition theorem $\mu_{w_e} = \mu_{v_e}^{\parallel} + \mu_{v_e}^{\perp}$ where, $\mu_{v_e}^{\parallel} << \mu_{v_e}$ and $\mu_{v_e}^{\perp} \perp \mu_{v_e}$. $\mu_{v_e}^{\parallel} \neq 0$ if and only if there is a choice of $v' \models p$ and $w' \models q$ such that $\tilde{H}_{v_e'} \cap \tilde{H}_{w_e'} \neq \{0\}$ and therefore $\tilde{H}_{v_e'} \not\perp \tilde{H}_{w_e'}$. □

Corollary 4.7.2. *Let $G \subseteq \tilde{H}$ be small. Let $p, q \in S_1(G)$, let $v \models p$ and $w \models q$. Then, $p \perp_G^a q$ if and only if $\mu_{P_G^{\perp} v_e} \perp \mu_{P_G^{\perp} w_e}$*

Proof. Clear from Theorem 4.7.1. □

Corollary 4.7.3. *Let $G \subseteq H$ be small. Let $p, q \in S_1(G)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Clear from Corollary 4.7.2. □

Theorem 4.7.4. *Let $v, w \in \tilde{H}$. Then $\tilde{H}_v$ is isometrically isomorphic to a Hilbert subspace of $\tilde{H}_w$ if and only if $\mu_v << \mu_w$.*

Proof. By Radon Nikodim Theorem, if $\mu_u << \mu_v$ then $\tilde{H}_v$ is isometrically equivalent to a Hilbert subspace of $\tilde{H}_w$. For the converse, if $\tilde{H}_v$ is isometrically equivalent to a Hilbert subspace of $\tilde{H}_w$, then v can be represented in $L^2(\mathbb{R}, \mu_w)$ by some function, and therefore, $\mu_u << \mu_v$. □

Theorem 4.7.5. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \rhd_\emptyset q$ if and only if $\mu_{v_e} >> \mu_{w_e}$.*

Proof. Suppose $p \rhd_\emptyset q$. Suppose that v and w are such that if $v \mathop{\smile\!\!\!\!\!\mid}_\emptyset^* G$ then $w \mathop{\smile\!\!\!\!\!\mid}_\emptyset^* G$ for every $G \subseteq \tilde{H}$. Then for every G if $\tilde{H}_{v_e} \perp \tilde{H}_G$ then $\tilde{H}_{w_e} \perp \tilde{H}_G$. This means $\tilde{H}_{w_e} \subseteq \tilde{H}_{v_e}$ and $\tilde{H}_{w_e}$ is unitarily equivalent to some Hilbert subspace of $\tilde{H}_{w_e}$ and by Theorem 4.7.4 $\mu_{w_e} << \mu_{v_e}$. □

Corollary 4.7.6. *Let E, F, and G be small subsets of $\tilde{H}$ and $p \in S_1(F)$ and $q \in S_1(G)$ two stationary types. Then $p \rhd_E q$ if and only if there exist v $w \in \tilde{\mathcal{H}}$ such that $gatp(v/E)$ is a non-forking extension of p, $gatp(w/E)$ is a non-forking extension of q and $\mu_{P_{acl(F)}^{\perp} v} >> \mu_{P_{acl(F)}^{\perp} w}$.*

Proof. Clear from previous theorem. □

4.8 Example: The Hydrogen Atom

The stationary states of time independent quantum systems are described by *Schrodinger's equation* (see [6])

$\mathfrak{H}\psi = E\psi$

Where $\mathfrak{H}$ denotes the *Hamiltonian operator of the system* and E denotes the energy of the stationary state.
For an hydrogen atom, this equation becomes:

$$E\psi = -\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{e^2}{4\pi\varepsilon_0 r}\psi$$

where, e is the electron charge, r is the distance of the electron to the nucleous of the atom and ε_0 is the electric permittivity of free space, and

$$\mu = \frac{m_e m_p}{m_e + m_p}$$

is the reduced mass of the proton-electron 2-body system. Here, m_e and m_p denote the rest mass of electron and proton respectively.
As we can see, this corresponds to find the spectrum of the operator $-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r}$ on the Hilbert space $L^2(\mathbb{R}^e)$ where $\mathbb{R}^3$ is provided with the usual Lebesgue measure. This operator is unbounded and self-adjoint, and its spectrum is given by:

$$\{-\frac{\hbar c R_\infty}{n^2} \mid n \in \mathbb{N}^+\} \cup [0, +\infty)$$

where, $R_\infty = \frac{m_e e^4}{8\varepsilon_0 \hbar c}$ is the so called *Rydberg constant*.
For $n \in \mathbb{N}^+$, the value $-\frac{\hbar c R_\infty}{n^2}$ is called the n^{th} *energy level*, and is denoted by E_n. Here the integer n is called the *quantum principal number*. In every energy level E_n, the stationary states are determined by other integers also calld quantum numbers. These quantum numbers are:

Orbital quantum number: Is denoted by ℓ and ranges as $\ell = 0, 1, \ldots, n-1$

Magnetic quantum number: Denoted by m_ℓ can vary as $m_\ell = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$. In total, they are $2\ell + 1$ possible values.

Spin quantum number: Is denoted by m_s can be $m_s = \frac{1}{2}, -\frac{1}{2}$

This quantum numbers depend on the momenta and other physical quantities which are not (Continuous First Order Logic) definable only from the Hamiltonian operator. So, our setting does not distiguish the differences between these secundary quantum numbers. So, the only we detect in our setting is the dimension of the eigenspace corresponding to the principal quantum number n has dimension which is $2\sum_{\ell=0}^{n}(2\ell + 1) = 4\sum_{\ell=0}^{n}\ell + 2 = 4\frac{n(n+1)}{2} + 2 = 2(n^2 + n + 1)$.
This shows us that all E_n are finite dimensional, so:

- $\sigma_d(\mathfrak{H}) = \{-\frac{\hbar c R_\infty}{n^2} \mid n \in \mathbb{N}^+\}$
- $\sigma_e(\mathfrak{H}) = [0, +\infty)$

So in our setting, (Galois) types of electrons (over the emptyset) are determined by the spectral measure on $\sigma(\mathfrak{H})$ defined by the electron, that is, the probability measure over the possible energy levels. The electrons that we are shure are trapped around the nucleous of the atom are the algebraic ones, whoose spectral measures have suport contained in $\{-\frac{\hbar c R_\infty}{n^2} \mid n \in \mathbb{N}^+\}$.
Other property that arises is that two electrons are independent if and only if their supports contained in $[0, \infty)$ are disjoint, that is, the probability of being in the same energy level is 0.
Finally, we can conclude that we can see the Schrodinger's equation as a dynamical system defined on a space of types, which coincide with the information that can be experimentally obtained.

5 Closed *-representations of *-algebras as metric abstract elementary classes

5.1 Introduction

We finish this thesis by trying to generalize many of the results given in Chapter 3 and Chapter 4. In other words, the goal of this last chapter is to track the properties of Hilbert spaces presented in the Introduction (Chapter 1) when we have not only one but a entire algebra of (not necessarily bounded) operators acting on H. This means that this chapter can be seen, as it was said before, as the "amalgam" of Chapter 3 and Chapter 4 over the basic model theoretic properties of Hilbert spaces.
More specificly, we deal with a Hilbert space under the action of a so called *O^*-algebra*, which is roughly speaking a generalization of a von Neumann algebra with the difference that operator in that algebra are unbounded.
The main results in this chapter are the following: Let $\mathcal{A}$ be a *-algebra and let $\pi : \mathcal{A} \to \mathcal{L}(D)$ a *-representation of $\mathcal{A}$ on H. Let us denote that structure by (H, D, π). We build a Metric Abstract Elementary Class (MAECS) associated with the structure (H, D, π) which is denoted by $\mathcal{K}_{(H,D,\pi)}$. Then we prove:

1. A characterization of (Galois) types of vectors, in terms of their associated functionals on the double commutant of the monster model of the class (see Theorem 5.3.4).

2. Let $\bar{v} \in H^n$ and $E \subseteq H$. Then the (Galois) type $gatp(\bar{v}/E)$ has a canonical base formed by a tuple of elements in H (see Theorem 5.5.11).

3. We characterize non-splitting in $\mathcal{K}_{(H,D,\pi)}$ and we show that it has the same properties as non-forking for superstable first order theories (see Theorem 5.5.7 and Theorem 5.5.8).

4. We characterize orthogonality and domination of types in terms of orthogonality and domination of their associated functionals on the double commutant of the monster model of the class (see Theorem 5.6.16 and Theorem 5.6.20).

This chapter is divided as follows: In Section 5.2, we define a *metric abstract elementary class* associated with (H, D, π) (denoted by $\mathcal{K}_{(H,D,\pi)}$) from a Voiculescu-like property on couples of *-representations. In Section 5.3, we characterize the Galois types of vectors, in terms of

their associated functionals on the double commutant of the monster model of the class. In Section 5.4, we give a characterization of definable and algebraic closures. In Section 5.5, we define independence in $\mathcal{K}_{(H,\Gamma_Q)}$ and we show that it is equivalent to non-splitting and has the same properties as non-forking for superstable first order theories. Finally in Section 5.6, we characterize domination, orthogonality of (Galois) types in terms of orthogonality and domination of their associated functionals on the double commutant of the monster model of the class.
The theoretical preliminaries about the theory of the *-algebras and the *-representations of them, needed in the rest of the chapter are in Section .5.

5.2 A metric abstract elementary class

Let D be a dense linear subspace of a Hilbert space H, $\mathcal{A}$ be a unital *-algebra and let $\pi : \mathcal{A} \to \mathcal{L}^\dagger(D)$ be a $*$-algebra nondegenerate closed representation (see Definition .5.24). A $\mathcal{A}$-structure is:

$(H, 0, +, i, (I_r)_{r\in\mathbb{Q}}, \|\cdot\|, (\Gamma_{\pi(a)})_{a\in\mathcal{A}})$

where

- H is a complex infinite dimensional Hilbert space
- 0 is the zero vector in H
- $+ : H \times H \to H$ is the usual sum of vectors in H
- $i : H \to H$ is the function that to any vector $v \in H$ assigns the vector iv where i is a complex number such that $i^2 = -1$.
- $I_r : H \to H$ is the function that sends every vector $v \in H$ to rv, where $r \in \mathbb{Q}$.
- $\|\cdot\| : H \to \mathbb{R}$ is the norm function
- Let $Q \in \pi(\mathcal{A})$, then $\Gamma_Q : H \times H \to \mathbb{R}$ is the function that to any v, $w \in H$ asigns the number $\Gamma_Q(v, w)$, which is the distance of (v, w) to the graph of Q. Since Q is closed, $\Gamma_Q(v, w) = 0$ if and only if (v, w) belongs to the graph of Q.

Briefly, the structure will be refered to either as (H, D, π).
In this section we build a Metric Abstract Elementary Class for the structure (H, D, π).

Definition 5.2.1. Let $\mathcal{A}$ be a *-algebra, let D be a dense subspace of a Hilbert space H and let $\pi : \mathcal{A} \to \mathcal{L}^\dagger(D)$ be a nondegenerate closed *-representation of $\mathcal{A}$. We define $\mathcal{K}_{(H,D,\pi)}$ to be the following class:

$$\mathcal{K}_{(H,D,\pi)} := \{(H', D', \pi') \mid H' \text{ is a Hilbert space, } D' \text{ is a dense linear subspace of } H'$$
$$\text{and } \pi' : \mathcal{A} \to \mathcal{L}^\dagger(D) \text{ is a } ^* \text{ representation such that } \forall a \in \mathcal{A}\ dim(\pi(a)D) = dim(\pi'(a)D')\}$$

We define the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,D,\pi)}$ by:

$(H_1, D_1, \pi_1) \prec_{\mathcal{K}} (H_2, D_2, \pi_2)$ if $H_1 \subseteq H_2,\ D_1 = D_2 \cap H_1,$ and for every $a \in \mathcal{A},\ \pi_1(a) \subseteq \pi_2(a)$

Remark 5.2.2. $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,D,\pi)}$ is trivial i.e. coincides with $\subseteq$

Lemma 5.2.3. *Let (H_1, D_1, π_1) and (H_2, D_2, π_2) be two $\mathcal{A}$-structures. An isomorphism $U : (H_1, D_1, \pi_1) \to (H_2, D_2, \pi_2)$ is a unitary operator of $U : H_1 \to H_2$ such that $UD(\pi_1(a)) = D(\pi_2(a))$ and $U\pi_1(a)v = \pi_2(a)Uv$ for every $a \in \mathcal{A}$ and $v \in D(\pi_1(a))$.*

Proof. $\Rightarrow$ Suppose U is an isomorphism between (H_1, D_1, π_1) and (H_2, D_2, π_2). It is clear that U must be a linear operator. Also, we have that for every $u,\ v \in \mathcal{H}$ we must have that $\langle Uu \,|\, Uv\rangle = \langle u \,|\, v\rangle$ by definition of automorphism. Therefore U must be an isometry and, therefore it must be unitary.

On the other hand, since U is an isomorphism between (H_1, D_1, π_1) and (H_2, D_2, π_2), for every $a \in \mathcal{A}$ and $(v, w) \in H_1 \times H_1$ we have that $\Gamma_{\pi_1(a)}(v, w) = \Gamma_{\pi_2(a)}(Uv, Uw)$. Therefore, for $a \in \mathcal{A}$, $\Gamma_{\pi_1(a)}(v, w) = 0$ if and only if $\Gamma_{\pi_2(a)}(Uv, Uw) = 0$. So, for every $v \in D(\pi_1(a))$, $U\pi_1(a)v = \pi_2(a)Uv$.

$\Leftarrow$ Let $U : H_1 \to H_2$ be an unitary operator such that $UD(\pi_1(a)) = D(\pi_2(a))$ and $U\pi_1(a)v = \pi_2(a)Uv$ for every $a \in \mathcal{A}$ and $v \in D(\pi_1(a))$. It remains to show that for every $a \in \mathcal{A}$ and for every $(v, w) \in H_1 \times H_1$, $\Gamma_{\pi_1(a)}(v, w) = \Gamma_{\pi_2(a)}(Uv, Uw)$. Let $a \in \mathcal{A}$ be a fixed element of $\mathcal{A}$, and let $(v, w) \in H \times H$ be any pair of vectors. There exists a sequence of pairs $(v_n, w_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $v_n \in D(\pi_1(a))$, $w_n = \pi_1(a)v_n$ and $\Gamma_{\pi_1(a)}(v, w) = \lim_{n\to\infty} d[(v, w); (v_n, w_n)]$.

By hypothesis, U is an isometry, and maps the graph of Q_1 into the graph of Q_2; so for all $n \in \mathbb{N}$, $Uv_n \in D(\pi_2(a))$ and $Uw_n = \pi_2(a)v_n$. We have that

$$\lim_{n\to\infty} d[(Uv, Uw); (Uv_n, Uw_n)] = \lim_{n\to\infty} d[(v, w); (v_n, w_n)] = \Gamma_{\pi_1(a)}(v, w).$$

So $\Gamma_{\pi_2(a)}(Uv, Uw) \leq \Gamma_{\pi(a)_1}(v, w)$. Repeating the argument for U^{-1}, we get $\Gamma_{\pi_1(a)}(v, w) \leq \Gamma_{\pi_2(a)}(Uv, Uw)$.

□

Corollary 5.2.4. *Let $\mathcal{A}$ be a *-algebra and let (H, D, π) be an $\mathcal{A}$-structure. Then $Aut(H, D, \pi)$ is the unitary group of $\pi(\mathcal{A})'$ (see Definition .5.36).*

Definition 5.2.5. Let (H, D, π) be a *-representation of $\mathcal{A}$. We define:

The essential part of π It is the C^*-algebra homomorphism,

$$\pi_e := \rho \circ \pi : \mathcal{A} \to \mathcal{L}(D)/\mathcal{V}(D)$$

of $\pi(\mathcal{A})$, where ρ is the canonical proyection of $\mathcal{L}(D)$ onto the Generalized Calkin Algebra $\mathcal{L}(D)/\mathcal{V}(D)$.

The discrete part of π It is the restriction,

$$\begin{aligned}\pi_d : ker(\pi_e) &\to \mathcal{Q}(D)\\ a &\to \pi(a)\end{aligned}$$

The discrete part of $\pi(\mathcal{A})$ It is defined in the following way:

$$\pi(\mathcal{A})_d := \pi(\mathcal{A}) \cap \mathcal{V}(D).$$

The essential part of $\pi(\mathcal{A})$ It is the image $\pi(\mathcal{A})_e$ of $\pi(\mathcal{A})$ in the Generalized Calkin Algebra.

The essential part of D It is defined in the following way:

$$D_e := \ker(\pi(\mathcal{A})_d)$$

The essential part of H It is defined in the following way:

$$H_e := \langle D_e \rangle$$

The discrete part of D It is defined in the following way:

$$D_d := \ker(D_e)^{\perp}$$

The discrete part of H It is defined in the following way:

$$H_d := \langle D_d \rangle$$

The essential part of a vector $v \in H$ It is the projection v_e of v over H_e.

The discrete part of a vector $v \in H$ It is the projection v_d of v over H_d.

The essential part of a set $E \subseteq H$ It is the set

$$E_e := \{v_e \mid v \in E\}$$

The discrete part of a set $G \subseteq H$ It is the set

$$E_d := \{v_d \mid v \in G\}$$

Definition 5.2.6. Let (H, D, π) be *-representation. Given $E \subseteq H$ and $v \in H$, we denote by:

1. $D_E := \langle E \rangle \cap D$, where $\langle E \rangle$ is the closed Hilbert subspace of H generated by E.

2. H_E, the Hilbert subspace of H generated by the elements $\pi(a)v$, where $v \in D_E$ and $a \in \mathcal{A}$.

3. $\pi_E := \pi \restriction H_E$.

4. (H_E, D_E, π_E), the subrepresentation of (H, D, π) generated by E.

5. H_v, the space H_E when $E = \{v\}$ for some vector $v \in H$

6. $\pi_v := \pi_E$ when $E = \{v\}$.

7. (H_v, D_v, π_v), the subrepresentation of (H, D, π) generated by v.

8. $D_E^{\perp}$, the othogonal complement of D_E in D.

9. $H_E^{\perp}$, the orthogonal complement of H_E in H

10. P_E, the projection over H_E.

11. $P_{E^{\perp}}$, the projection over $H_E^{\perp}$.

Theorem 5.2.7. *Let $\mathcal{A}$ be a *-algebra, let D be a dense subspace of a Hilbert space H and let $\pi : \mathcal{A} \to \mathcal{L}^{\dagger}(D)$ be a nondegenerate closed *-representation of $\mathcal{A}$. Then $\mathcal{K}_{(H,D,\pi)}$ is a metric abstract elementary class.*

Proof. 1. Closure under isomorphism:

a) Let $(H_1, D_1, \pi_1) \simeq (H_2, D_2, \pi_2) \in \mathcal{K}_{(H,D,\pi)}$. By Lemma 5.2.3, this means that there is a unitary operator $U : H_1 \to H_2$ such that $UD(\pi_1(a)) = D(\pi_2(a))$ and $U\pi_1(a)v = \pi_2(a)Uv$ for every $a \in \mathcal{A}$ and $v \in D(\pi_1(a))$. This implies that $\forall a \in \mathcal{A}\ \ dim(\pi(a)D_1) = dim(\pi'(a)D_2) = dim(\pi'(a)D)$. So, $(H_1, D_1, \pi_1) \in \mathcal{K}_{(H,D,\pi)}$.

b) Clear since by Remark 5.2.2 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,D,\pi)}$.

2. Clear since by Remark 5.2.2 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,D,\pi)}$.

3. Clear since by Remark 5.2.2 $\prec_{\mathcal{K}}$ is trivial in $\mathcal{K}_{(H,D,\pi)}$.

4. $LS(\mathcal{K}_{(H,D,\pi)}) \leq \aleph_0 + |\mathcal{A}|$. Let $(H, D, \pi) \in \mathcal{K}_{(H,D,\pi)}$ and $G \subseteq H$. Since D is dense in H, for every $v \in G$ there is a sequence $(v_n)_{n\in\mathbb{N}} \subseteq D$ such that $v_n \to v$. Let $G_0 := \cup_{v\in G}\{v_n \mid n \in \mathbb{N}\}$. For $n \in \mathbb{N}$, let $G_{n+1} := \pi(\mathcal{A})G_n$, and let $G' := \cup_{n=0}^{\infty} G_n$. Then, we have that $G' \subseteq D$, $\pi(\mathcal{A})G' \subseteq G'$. Let H' be the completion of G' in H and let $p' := \pi \restriction G'$. Then (H', G', π') is a $*$-representation of $\mathcal{A}$ such that $\forall a \in \mathcal{A}\ \ dim(\pi(a)D) = dim(\pi'(a)D')$. So, $(H', G', \pi') \in \mathcal{K}_{(H,D,\pi)}$. Since $\text{dens}(H') \leq |G'|$ and $|G'| \leq |G| + \aleph_0 + |\mathcal{A}|$, we have that $\text{dens}(H') \leq |G| + \aleph_0 + |\mathcal{A}|$.

5. Tarski-Vaught chain:

a) Suppose κ is an infinite cardinal and $(\hat{H}, \hat{\pi}, \hat{D}) \in \mathcal{K}_{(H,D,\pi)}$. Let $(H_i, \pi_i, D_i)_{i<\kappa}$ a $\prec_{\mathcal{K}_{(H,D,\pi)}}$ increasing sequence such that $(H_i, \pi_i, D_i) \prec_{\mathcal{K}_{(H,D,\pi)}} (\hat{H}, \hat{\pi}, \hat{D})$ for all $i < \kappa$. We have that for each $i < \kappa$, $(H_i, \pi_i, D_i) = (H_d, D_d, \pi_d) \oplus \big((H_i)_e, (D_i)_e, (\pi_i)_e\big)$. Let $H' := \overline{\cup_{n=0}^{\infty} D_n}$, $D' := \cup_{n=0}^{\infty} D_n$ and $\pi' := \cup_{n=0}^{\infty} \hat{\pi} \restriction D_n$. Then $(H', D', \pi') = (H_d, D_d, \pi_d) \oplus \bigoplus_{i<\kappa} \big((H_i)_e, (D_i)_e, (\pi_i)_e\big)$. For every $v \in (D_i)_e$ and for all $a \in \mathcal{A}$, $dim((\pi_i)_e(D_i)_e = \infty$. So, for $a \in \mathcal{A}$, $dim(\pi'(a)D') = \sup_n dim(\pi_n(a)D_n)$. But for every $a \in \mathcal{A}$, the sequence $dim(\pi_n(a)D_n)$ is constant equal to $dim(\pi(a)D)$, so $dim(\pi'(a)D') = dim(\pi(a)D)$. Then, $(H', D', \pi') \in \mathcal{K}_{(H,D,\pi)}$ and $(H, D, \pi) \prec_{\mathcal{K}_{(H,D,\pi)}} (\hat{H}, \hat{D}, \hat{\pi})$.

b) Clear from previous item.

□

Theorem 5.2.8. *Let $\mathcal{A}$ be a *-algebra, let D be a dense subspace of a Hilbert space H and let $\pi : \mathcal{A} \to \mathcal{L}^{\dagger}(D)$ be a nondegenerate closed *-representation of $\mathcal{A}$. $\mathcal{K}_{(H,D,\pi)}$ has the AP.*

Proof. Let (H_1, D_1, π_1), (H_2, D_2, π_2) and $(H_3, D_3, \pi_3) \in \mathcal{K}_{(H,D,\pi)}$ be such that $(H_1, D_1, \pi_1) \prec (H_2, D_2, \pi_2)$ and $(H_1, D_1, \pi_1) \prec (H_3, D_3, \pi_3)$. Since H_2 and D_2 are complex vector spaces with an inner product, and D_1 is closed in D_2 there are a Hilbert space H', and a complex vector space D' such that:

- $H_2 = H_1 \oplus H'$
- $D_2 = D_1 \oplus D'$
- $\pi_2 = \pi_1 \oplus (\pi_2 \restriction D')$

Let

$H_4 := H_3 \oplus H'$,

$D_4 := D_3 \oplus D'$

and

$\pi_4 := \pi_3 \oplus (\pi_2 \restriction D')$

Then $(H_4, D_4, \pi_4) \in \mathcal{K}_{(H,D,\pi)}$, $(H_2, D_2, \pi_2) \prec (H_4, D_4, \pi_4)$ and $(H_3, D_3, \pi_3) \prec (H_4, D_4, \pi_4)$.

□

Theorem 5.2.9. *Let $\mathcal{A}$ be a *-algebra, let D be a dense subspace of a Hilbert space H and let $\pi : \mathcal{A} \to \mathcal{L}^{\dagger}(D)$ be a nondegenerate closed *-representation of $\mathcal{A}$. $\mathcal{K}_{(H,D,\pi)}$ has the JEP.*

Proof. Let (H_1, D_1, π_1), $(H_2, D_2, \pi_2) \in \mathcal{K}_{(H,D,\pi)}$ and. Let

- $H_3 = H_1 \oplus H_2$

- $D_3 = D_1 \oplus D_2$
- $\pi_3 = \pi_1 \oplus \pi_2$

Then $(H_3, D_3, \pi_3) \in \mathcal{K}_{(H,D,\pi)}$, $(H_1, D_1, \pi_1) \prec (H_3, D_3, \pi_3)$ and $(H_2, D_2, \pi_2) \prec (H_3, D_3, \pi_3)$. □

Remark 5.2.10. Recall that if $(H_i, D_i, \pi_i)_{i\in I}$ is an indexed family of $\mathcal{A}$-structures, we can define $\bigoplus_{i\in I}(H_i, D_i, \pi_i)$ in the way described in Definition .5.29.

Theorem 5.2.11. *Let $\mathcal{A}$ be a *-algebra, let D be a dense subspace of a Hilbert space H and let $\pi : \mathcal{A} \to \mathcal{L}^\dagger(D)$ be a nondegenerate closed *-representation of $\mathcal{A}$. For every cardinal κ, $\mathcal{K}_{(H,D,\pi)}$ has structures wit density $\geq \kappa$.*

Proof. For κ an infinite cardinal, it is clear that the structre

$$(H_d, D_d, \pi_d) \oplus \bigoplus_{\kappa}(H_{S_{\pi(\mathcal{A})_e}}, D_{S_{\pi(\mathcal{A})_e}}, \pi_{S_{\pi(\mathcal{A})_e}})$$

belongs to $\mathcal{K}_{(H,D,\pi)}$ and has density $\geq \kappa$. □

Fact 5.2.12 (Remark 2.6 in [35])**.** If $\mathcal{K}$ is a MAEC which satisfies AP and JEP and has large enough models, then we can construct a large model $\tilde{\mathcal{M}}$ called *monster model* which is homogeneous -i.e., every isomorphism between small substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$- and also universal -i.e., every model with density $< dc(\mathcal{M})$ can be embedded in $\mathcal{M}$.

Corollary 5.2.13. *The class $\mathcal{K}_{(H,D,\pi)}$ has a monster model which, from now on, $(\tilde{H}, \tilde{D}, \tilde{\pi})$ will be denoted by $\mathcal{K}_{(H,D,\pi)}$.*

5.3 Types

In this section, we characterize of (Galois) types of vectors, in terms of their associated functionals on the double commutant of the monster model of the class.

Definition 5.3.1. For $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$, and $(a_i)_{i<\alpha} subseteq \mathcal{M}_1$, $(b_i)_{i<\alpha} \subseteq \mathcal{M}_2$, we say that $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ have the same *Galois type* in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively,

$$gatp_{\mathcal{M}_1}((a_i)_{i<\alpha}/\emptyset) = gatp_{\mathcal{M}_2}((b_i)_{i<\alpha}/\emptyset),$$

if there are $\mathcal{N} \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$ such that $(a_i) = g(b_i)$ for every $i < \alpha$.

Remark 5.3.2. Under the existence of a monster model, to say $(gatp_{\mathcal{M}_1}((a_i)_{i<\alpha}/A) = gatp_{\mathcal{M}_2}((b_i)_{i<\alpha}/A))$ is the same as stating that there exists an automorphism U of the monster model such that for all $i > \alpha$, $Ub_i = a_i$ and for all $a \in A$, $Ua = a$.

Theorem 5.3.3. *Let $(\hat{H}, \hat{D}, \hat{\pi})$ be the monster model structure of $\mathcal{K}_{(H,D,\pi)}$. Let v, $w \in \hat{D}$. Then, $gatp(v/\emptyset) = gatp(w/\emptyset)$ if and only if for every $a \in \mathcal{A}$, $\phi_v(a) := \langle \pi(a)v|v\rangle = \langle \pi(a)w|w\rangle =: \phi_w$.*

Proof. $\Rightarrow$ Let (H_1, D_1, π_1) and $(H_2, D_2, \pi_2) \in \mathcal{K}_{(H,D,\pi)}$, and suppose $gatp(v/\emptyset) = gatp(w/\emptyset)$ for $v \in D_1$ and $w \in D_2$. Then there exists $(H_3, D_3, \pi_3) \in \mathcal{K}_{(H,D,\pi)}$ and $\mathcal{K}_{(H,D,\pi)}$-embeddings $U_1 : (H_1, D_1, \pi_1) \to (H_3, D_3, \pi_3)$ and $U_1 : (H_2, D_2, \pi_2) \to (H_3, D_3, \pi_3)$ such that $U_1 v = U_2 w$. Since U_1 and U_2 are $\mathcal{K}_{(H,D,\pi)}$-embeddings, $\phi_v = \phi_{U_1 v} = \phi_{U_2 w} = \phi_w$.

$\Leftarrow$ Suppose $\phi_v = \phi_w$. Then Let (H_3, D_3, π_3) be the amalgamation of (H_1, D_1, π_1) and (H_2, D_2, π_2) over the GNS construction of $\phi = \phi_v = \phi_w$. Then, it is clear that there are $\mathcal{K}_{(H,D,\pi)}$-embedings $U_1 : (H_1, D_1, \pi_1) \to (H_3, D_3, \pi_3)$ and $U_3 : (H_2, D_2, \pi_2) \to (H_3, D_3, \pi_3)$ such that $U_1 v = v_\phi = U_2 w$.

□

Theorem 5.3.4. *Let $(\hat{H}, \hat{D}, \hat{\pi})$ be the monster model structure of $\mathcal{K}_{(H,D,\pi)}$. Let v, $w \in \hat{H}$. Then, $gatp(v/\emptyset) = gatp(w/\emptyset)$, if and only if $\phi_v = \phi_w$, where ϕ_v and ϕ_w are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_v(S) = \langle Sv|v\rangle$ and $\phi_w(S) = \langle Sw|w\rangle$.*

Proof. $\Rightarrow$ Suppose such that $gatp(v/\emptyset) = gatp(w/\emptyset)$. Then, there exists an automorphism U of $(\hat{H}, \hat{D}, \hat{\pi})$ such that $Uv = w$. By Lemma 5.2.3, U commutes with every $\hat{\pi}(a)$ with $a \in \mathcal{A}$ and therefore $U \in \hat{\pi}(\mathcal{A})'_w$. Let $S \in \hat{\pi}(\mathcal{A})''_w$. Then $\langle Sv \mid v\rangle = \langle USv \mid Uv\rangle = \langle SUv \mid Uv\rangle = \langle Sw \mid w\rangle$.

$\Leftarrow$ Suppose $\phi_v = \phi_w$ on $\hat{\pi}(\mathcal{A})''_w$. By Theorem 3.4.4, v and w have the same type in H when H is seen as a representation of $\hat{\pi}(\mathcal{A})''_w$. So, there is an automorphism $U \in \hat{\pi}(\mathcal{A})'''_w = \hat{\pi}(\mathcal{A})'_w$ such that $Uv = w$. But this means is an automorphism of $(\hat{H}, \hat{D}, \hat{\pi})$ as a *-representation of $\mathcal{A}$.

□

Theorem 5.3.5. *Let $(\hat{H}, \hat{D}, \hat{\pi})$ be the monster model structure of $\mathcal{K}_{(H,D,\pi)}$. Let v, $w \in \hat{H}$. Then, $gatp(v/\emptyset) = gatp(w/\emptyset)$, if and only if $(H_v, D_v, \pi_v) \simeq (H_w, D_w, \pi_w)$.*

Proof. $\Rightarrow$ Let v, $w \in \hat{H}$ be such that $gatp(v/\emptyset) = gatp(w/\emptyset)$. Then, there exists an automorphism U of $(\hat{H}, \hat{D}, \hat{\pi})$ such that $Uv = w$. So D_v and D_w are isomorphic as well as H_v and H_w. Therefore $(H_v, D_v, \pi_v) \simeq (H_w, D_w, \pi_w)$.

$\Leftarrow$ Suppose now that $(H_v, D_v, \pi_v) \simeq (H_w, D_w, \pi_w)$. Then, $(H_d, D_d, \pi_d) \oplus (H_{v_e}, D_{v_e}, \pi_{v_e}) \simeq (H_d, D_d, \pi_d) \oplus (H_{w_e}, D_{w_e}, \pi_{w_e})$. But $(H_d, D_d, \pi_d) \oplus (H_{v_e}, D_{v_e}, \pi_{v_e}) \in \mathcal{K}_{(H,D,\pi)}$ and $(H_d, D_d, \pi_d) \oplus (H_{w_e}, D_{w_e}, \pi_{w_e}) \in \mathcal{K}_{(H,D,\pi)}$. Since $\mathcal{K}_{(H,D,\pi)}$ is an homogeneous class, the isomorphism between this structures can be extended to an automorphism of $(\hat{H}, \hat{D}, \hat{\pi})$.

□

Theorem 5.3.6. *Let $v \in (H_1, D_1, \pi_1) \in \mathcal{K}_{(H,D,\pi)}$, $w \in\in (H_2, D_2, \pi_2) \in \mathcal{K}_{(H,D,\pi)}$ and $G \subseteq H_1 \cap H_2$ such that $(H_G, D_G, \pi_G) \in \mathcal{K}_{(H,D,\pi)}$, $(H_G, D_G, \pi_G) \prec (H_1, D_1, \pi_1)$ and $(H_G, D_G, \pi_G) \prec (H_2, D_2, \pi_2)$. Then $gatp_{(H_1,D_1,\pi_1)}(v/G) = gatp_{(H_2,D_2,\pi_2)}(w/G)$ if and only if*

$P_G v = P_G w$

and

$gatp(P_G^{\perp}(v)/\emptyset) = gatp(P_G^{\perp}(w)/\emptyset)$

Proof. $\Rightarrow$ Suppose $gatp_{(H_1,D_1,\pi_1)}(v/G) = gatp_{(H_2,D_2,\pi_2)}(w/G)$ and let $v' := P_{G^\perp} v$ and $w' := P_{G^\perp} w$. Then, by Definition 4.2.20, there exists $(H_3, D_3, \pi_3) \in \mathcal{K}_{(H,D,\pi)}$ and $\mathcal{K}_{(H,D,\pi)}$-embeddings $U_1 : (H_1, D_1, \pi_1) \to (H_3, D_3, \pi_3)$ and $U_2 : (H_2, D_2, \pi_2) \to (H_3, D_3, \pi_3)$ such that $U_1 v = U_2 w$ and $U_1 \restriction G \equiv U_2 \restriction G \equiv Id_G$, where Id_G is the identity on G. Since $v = P_G v + P_{G^\perp} v$ and $w = P_G w + P_{G^\perp} w$ we have that $U_1 P_G v = PG_v = U_2 P_G w = P_G w$. Since U_1 and U_2 are embeddings, $gatp(v'/\emptyset) = gatp(U_1 v'/\emptyset) = gatp(U_2 w'/\emptyset) = gatp(w'/\emptyset)$

$\Leftarrow$ Suppose $P_G v = P_G w$ and $gatp(v'/\emptyset) = gatp(w'/\emptyset)$, where $v' = P_{G^\perp} v$ and $w' = P_{G^\perp} w$. Then, $gatp(v'_e/\emptyset) = gatp(w'_e/\emptyset)$ and $(H_{v_e}, D_{v_e}, \pi_{v_e}) \simeq (H_{w_e}, D_{w_e}, \pi_{w_e})$. Let $(H', D', \pi') := (H_{v_e}, D_{v_e}, \pi_{v_e}) \simeq (H_{w_e}, D_{w_e}, \pi_{w_e})$. Also, let:

$\hat{H} := (H_1 \vee_{H_G} H_2) \oplus H'$,

$\hat{D} := (D_1 \vee_{D_G} D_2) \oplus D'$,

and

$\hat{\pi} := (\pi_1 \vee_{\pi_G}) \pi_2 \oplus \pi'$.

Let $U_1 : (H_1, D_1, \pi_1) \to (\hat{H}, \hat{D}, \hat{\pi})$ be the $\mathcal{K}_{(H,D,\pi)}$-embedding acting on $H_{v'}^{\perp} \vee H_{w'}^{\perp}$ as in the AP. Define $U_2 : (H_2, D_2, \pi_2) \to (\hat{H}, \hat{D}, \hat{\pi})$ in the same way. Then, we have completed the conditions to show that $gatp_{(H_1,D_1,\pi_1)}(v/G) = gatp_{(H_2,D_2,\pi_2)}(w/G)$. □

Theorem 5.3.7. *$\mathcal{K}_{(H,D,\pi)}$ has the continuity of types property*

Proof. Let $G \subseteq \tilde{H}$ be small and $(v_i)_{i<\omega} \subseteq \tilde{H}$ be a sequence such that $\lim_{i\to\infty} v_i = v$ and $gatp(v_i/A) = gatp(v_j/A)$ for all i, $j < \omega$. Then, by Theorem 5.3.6, $P_G v_i = P_G v_j$ and $gatp(P_G^{\perp}(v_i)/\emptyset) = gatp(P_G^{\perp}(v_j)/\emptyset)$ for all i, $j < \omega$. If $\lim_{i\to\infty} v_i = v$, it is clear that $P_G v_i = P_G v$ for all $i < \omega$. So, it is enough to prove the theorem for the case $G = \emptyset$.
Suppose $\lim_{i\to\infty} v_i = v$ and $gatp(v_i/\emptyset) = gatp(v_j/\emptyset)$ for all i, $j < \omega$. By Theorem 5.3.5, thsi means that $(H_{v_i}, D_{v_i}, \pi_{v_i}) \simeq (H_{v_j}, D_{v_j}, \pi_{v_j})$. Let (H', D', π') be any of these $(H_{v_i}, D_{v_i}, \pi_{v_i})$. If $\lim_{i\to\infty} v_i = v$, we can asume that $v_i \in (H_v, D_v, \pi_v)$ for all $i < \omega$. So, (H', D', π') can be embedded into (H_v, D_v, π_v). If (H_v, D_v, π_v) couldn't be embedded into (H', D', π') there would exists $w \in H_v$ such that $H_w \perp H^{\perp}$. So, $w \perp v_i$ for all $i < \omega$. Since $w \in H_v$, this implies that $\lim_{i\to\infty} v_i \neq v$, which is a contradiction. □

5.4 Definable and algebraic closures

Lemma 5.4.1. *Let $E \subseteq H$, $U \in Aut(H, D, \pi)$. Then $U \in Aut((H, D, \pi)/E)$ if and only if $U \restriction (H_E, D_E, \pi_E) = Id_{(H_E,D_E,\pi_E)}$*

Proof. Suppose that $U \restriction (H_E, D_E, \pi_E) = Id_{(H_E,D_E,\pi_E)}$. Then U fixes H_E pointwise, and therefore, fixes E pointwise. Conversely, suppose $U \in Aut((H, D, \pi)/E)$. By Remark 5.2.3 U is an unitary operator that commutes with every $S \in \pi(\mathcal{A})$. Then for every $S \in \pi(\mathcal{A})$ and $v \in D_E$, we have that $U(Sv) = S(Uv) = Sv$. So U acts on (H_E, D_E, π_E) and the conclusion follows. □

Theorem 5.4.2. *Let $E \subseteq H$. Then $ga - dcl(\emptyset) = H_E$*

Proof. From Lemma 5.4.1, it is clear that $H_E \subseteq ga - dcl(E)$. On the other hand, if $v \in H_E$, let $\lambda \in \mathcal{C}$ such that $\lambda \neq 1$ and $|\lambda| = 1$. Then, the operator $U := Id_{(H_E,D_E,\pi_E)} \oplus Id_{(H_E^{\perp},D_E^{\perp},\pi_E^{\perp})}$ is an automorphism of (H, D, π) fixing E such that $Uv \neq v$. □

Lemma 5.4.3. *Let $v \in H_e$. Then v is not algebraic over $\emptyset$.*

Proof. We can asume that $(\tilde{H}, \tilde{D}, \tilde{\pi})$ is the monster model density $\kappa > 2^{\aleph_0}$. Then there are κ vectors v_i for $i < \kappa$ such that every v_i has the same type over $\emptyset$ as v. This means that the orbit of v under the automorphisms of $(\tilde{H}, \tilde{D}, \tilde{\pi})$ is unbounded and therefore v is not algebraic over the emptyset. □

Lemma 5.4.4. *Let $v \in H$ such that $v_e \neq 0$. Then v is not algebraic over $\emptyset$.*

Proof. Clear from previous Lemma 5.4.3. □

Lemma 5.4.5. *Let $v \in H_d$. Then v is algebraic over $\emptyset$*

Proof. If $v \in H_d$ by Theorem .5.41 then v is either null or there exists an $S \in \pi(\mathcal{A}) \cap \mathcal{V}(D)$ and a sequence $v_i \in D$ such that $v_i \to v$ when $i \to \infty$ and $Sv_i \neq 0$. But in that case $(v_i)_{i<\omega}$ is a bounded set in $D_{\pi(\mathcal{A})}$ so $(Sv_i)_{i<\omega}$ is relatively compact int hte graph topology defined by $\pi(\mathcal{A})$ in D. So, the orbit of v under any automorphism U of (H, D, π) is compact, which implies that v is algebraic. □

Theorem 5.4.6. *$ga - acl(\emptyset) = H_d$.*

Proof. By Lemma 5.4.5, $H_d \subseteq ga - acl(\emptyset)$ and, by Lemma 5.4.4, $ga - acl(\emptyset) \subseteq H_d$ □

Theorem 5.4.7. *Let $E \subseteq H$. Then $ga - acl(E)$ is the Hilbert subspace of H generated by $ga - dcl(E)$ and $ga - acl(\emptyset)$.*

Proof. Let G be the Hilbert subspace of H generated by $ga - dcl(E)$ and $ga - acl(\emptyset)$. It is clear that $G \subseteq ga - acl(E)$. Let $v \in ga - acl(E)$. By Lemma 5.4.5, $v_d \in ga - acl(\emptyset)$, and by Theorem 5.4.2 and Lemma 5.4.3, $v_e \in ga - dcl(E) \setminus ga - acl(\emptyset)$. Then $v_e \in ga - dcl(E)$ and $ga - acl(E) \subseteq G$. □

5.5 Independence and splitting

In this section, we characterize non-splitting in $\mathcal{K}_{(H,D,\pi)}$ and we show that it has the same properties as non-forking for superstable first order theories.

Definition 5.5.1. Let E, F, $G \subseteq H$. We say that E is *independent* from G over F if for all $v \in E$ $P_{ga-acl(F)}(v) = P_{ga-acl(F \cup G)}(v)$ and denote it by $E \downarrow^*_F G$.

Remark 5.5.2. Let $\bar{v}$, $\bar{w} \in H^n$ and $E \subseteq H$. Then it is easy to see that:

- $\bar{v}$ is independent from $\bar{w}$ over $\emptyset$ if and only if for every $j, k = 1, \ldots, n$, $H_{(v_j)_e} \perp H_{(w_k)_e}$.
- $\bar{v}$ is independent from $\bar{w}$ over E if and only if for every $j, k = 1, \ldots, n$, $H_{P_E^\perp(v_j)_e} \perp H_{P_E^\perp(w_k)_e}$.
- $\bar{v} \in H^n$ and E, $F \subseteq H$. Then $\bar{v} \downarrow^*_E F$ if and only if for every $j = 1, \ldots, n$ $v_j \downarrow^*_E F$ that is, for all $j = 1, \ldots, n$ $P_{ga-acl(E)}(v_j) = P_{ga-acl(E \cup F)}(v_j)$.

Theorem 5.5.3. *Let $E \subseteq F \subseteq H$, $p \in S_n(E)$, $q \in S_n(F)$ and $\bar{v} = (v_1, \ldots, v_n)$, $\bar{w} = (w_1, \ldots, w_n) \in H^n$ be such that $p = gatp(\bar{v}/E)$ and $q = gatp(\bar{w}/F)$. Then q is an extension of p such that $\bar{w} \downarrow^*_E F$ if and only if the following conditions hold:*

1. *For every $j = 1, \ldots, n$, $P_{ga-acl(E)}(v_j) = P_{ga-acl(F)}(w_j)$*
2. *For every $j = 1, \ldots, n$, $(H_{P^\perp_{ga-acl(E)} v_j}, D_{P^\perp_{ga-acl(E)} v_j}, \pi_{P^\perp_{ga-acl(E)} v_j})$ is isometrically isomorphic to $(H_{P^\perp_{ga-acl(F)} w_j}, D_{P^\perp_{ga-acl(F)} w_j}, \pi_{P^\perp_{ga-acl(F)} w_j})$*

Proof. Clear from Theorem 5.3.3 and Remark 5.5.2 □

Remark 5.5.4. Recall that for every $E \subseteq H$ and $v \in H$, $P^\perp_{ga-acl(E)} v = (P^\perp_E v)_e$.

Theorem 5.5.5. $\downarrow^*$ *is a freeness relation.*

Proof. By Remark 5.5.2, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character Let $v \in H$ and $E \subseteq H$. Let $w = (P_{ga-acl(E)}(v))_e$. Then there exist a sequence of $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, a sequence of finite tuples $(a_1^k, \ldots, a_{l_k}^k)_{k \in \mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e_1^k, \ldots, e_{l_k}^k)_{k \in \mathbb{N}} \subseteq E$ such that if $w_k := \sum_{j=1}^{l_k} \pi(a_j^k) e_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$. Let $E_0 = \{e_j^k \mid j = 1, \ldots, l_k \text{ and } k \in \mathbb{N}\}$. Then $v \downarrow^*_{E_0} E$ and $|E_0| = \aleph_0$.

Finite character We show that for $v \in H$, $E, F \subseteq H$, $v \downarrow^*_E F$ if and only if $v \downarrow^*_E F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\downarrow^*_E F$. Let $w = P_{ga-acl(E \cup F)}(v) - P_{ga-acl(E)}(v)$. Then $w \in \text{acl}(E \cup F) \setminus \text{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}^2$, a sequence of finite tuples $(a_1^k, \dots, a_{l_k+n_k}^k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k, f_1^k, \dots, f_{n_k}^k)_{k\in\mathbb{N}}$ such that $(e_1^k, \dots, e_{l_k}^k) \subseteq E$, $(f_1^k, \dots, f_{n_k}^k)_{k\in\mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} \pi(a_j^k)e_i^k + \sum_{j=1}^{n_k} \pi(a_{l_k+j}^k)f_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$.

Since $v \not\mathop{\smile\hspace{-0.9em}\mid}^*_E F$, then $w = P_{ga-acl(E\cup F)}(v) - P_{ga-acl(E)}(v) \neq 0$. Let $\epsilon = \|w\| > 0$. Then, there is k_ϵ such that if $k \geq k_\epsilon$ then $\|w - w_k\| < \epsilon$. Let $F_0 := \{f_1^1, \dots, f_{k_\epsilon}^{n_{k_\epsilon}}\}$, then F_0 is a finite subset such that $v \not\mathop{\smile\hspace{-0.9em}\mid}^*_E F_0$.

Transitivity of independence Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \mathop{\smile\hspace{-0.9em}\mid}^*_E G$ then $P_{ga-acl(E)}(v) = P_{ga-acl(G)}(v)$. It is clear that $P_{ga-acl(E)}(v) = P_{ga-acl(F)}(v) = P_{ga-acl(G)}(v)$ so $v \mathop{\smile\hspace{-0.9em}\mid}^*_E F$ and $v \mathop{\smile\hspace{-0.9em}\mid}^*_F G$. Conversely, if $v \mathop{\smile\hspace{-0.9em}\mid}^*_E F$ and $v \mathop{\smile\hspace{-0.9em}\mid}^*_F G$, we have that $P_{ga-acl(E)}(v) = P_{ga-acl(F)}(v)$ and $P_{ga-acl(F)}(v) = P_{ga-acl(G)}(v)$. Then $P_{ga-acl(E)}(v) = P_{ga-acl(G)}(v)$ and $v \mathop{\smile\hspace{-0.9em}\mid}^*_E G$.

Symmetry It is clear from Remark 5.5.2.

Invariance Let U be an automorphism of (H, D, π). Let $\bar{v} = (v_1, \dots, v_n)$,$\bar{w} = (w_1, \dots, w_n) \in H^n$ and $E \subseteq H$ be such that $\bar{v} \mathop{\smile\hspace{-0.9em}\mid}^*_E \bar{w}$. By Remark 5.5.2, this means that for every j, $k = 1, \dots, n$ $H_{P^\perp_{ga-acl(E)}(v_j)} \perp H_{P^\perp_{ga-acl(E)}(w_k)}$. It follows that for every j, $k = 1, \dots, n$ $H_{P^\perp_{ga-acl(UE)}(Uv_j)} \perp H_{P^\perp_{ga-acl(UE)}(Uw_k)}$ and, again by Remark 5.5.2, $Uv \mathop{\smile\hspace{-0.9em}\mid}^*_{ga-acl(UE)} Uw$.

Existence Let $(\tilde{H}, \tilde{\pi})$ be the monster model and let $E \subseteq F \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(E)$, there exists $q \in S_n(F)$ such that q is a non-forking extension of p.

Case $n = 1$ Let $v \in \tilde{H}$ be such that $p = gatp(v/E)$ and let $(H', \pi', u) := (H_{(P^\perp_{ga-acl(E)}v)}, \pi_{(P^\perp_{ga-acl(E)}v)}, P^\perp_{ga-}$
Then, by Fact 3.2.11, the model $(\hat{H}, \hat{D}, \hat{\pi}) := (H, D, \pi) \oplus (H', D', \pi')$ is an elementary extension of (H, D, π). Let $v' := P_{ga-acl(E)}v + P^\perp_{ga-acl(E)}v_d + u \in \hat{H}$. Then, by Theorem 5.5.3, the type $gatp(v'/F)$ is a $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of $gatp(v/E)$.

Induction step Now, let $\bar{v} = (v_1, \dots, v_n, v_{n+1}) \in \tilde{H}^{n+1}$. By induction hypothesis, there are $v_1', \dots, v_n' \in H$ such that $gatp(v_1', \dots, v_n'/F)$ is a $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of $gatp(v_1, \dots, v_n/E)$. Let U be an automorphism of the monster model fixing E pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v_j'$. Let $v_{n+1}' \in \tilde{H}$ be such that $gatp(v_{n+1}'/Fv_1' \cdots v_n')$ is a $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of $gatp(U(v_{n+1})/Ev_1', \cdots v_n')$. Then, by transitivity, $gatp(v_1', \dots, v_n', v_{n+1}'/F)$ is a $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of $gatp(v_1, \dots, v_n, v_{n+1}/E)$.

Stationarity Let $(\tilde{H}, \tilde{\pi})$ be the monster model and let $E \subseteq F \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(E)$, if $q \in S_n(F)$ is a $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of p to F then $q = p'$, where p' is the $\mathop{\smile\hspace{-0.9em}\mid}^*$-independent extension of p to F built in the proof of existence.

Case $n = 1$ Let $v \in H$ be such that $p = gatp(v/E)$, and let $q \in S(F)$ and $w \in H$ be such that $w \models q$. Let v' be as in previous item. Then, by Theorem 5.5.3 we have that:

1. $P_{ga-acl(E)}v = P_{ga-acl(F)}v' = P_{ga-acl(F)}w$
2. $(H_{P^{\perp}_{ga-acl(E)}v}, D_{P^{\perp}_{ga-acl(E)}v}, \pi_{P^{\perp}_{ga-acl(E)}v})$ is isometrically isomorphic to both

   $(H_{P^{\perp}_{ga-acl(F)}w}, D_{P^{\perp}_{ga-acl(F)}w}, \pi_{P^{\perp}_{ga-acl(F)}w})$

   and

   $(H_{P^{\perp}_{ga-acl(F)}v'}, D_{P^{\perp}_{ga-acl(F)}v'}, \pi_{P^{\perp}_{ga-acl(F)}v'})$

This means that $P_{ga-acl(F)}v' = P_{ga-acl(F)}w$ and $(H_{P^{\perp}_{ga-acl(F)}w}, D_{P^{\perp}_{ga-acl(F)}w}, \pi_{P^{\perp}_{ga-acl(F)}w})$ is isometrically isomorphic to $(H_{P^{\perp}_{ga-acl(F)}v'}, D_{P^{\perp}_{ga-acl(F)}v'}, \pi_{P^{\perp}_{ga-acl(F)}v'})$ and, therefore $q = gatp(v'/F) = p'$.

Induction step Let $\bar{v} = (v_1, \ldots, v_n, v_{n+1})$, $\bar{v}' = (v'_1, \ldots, v_n, v'_{n+1})$ and $\bar{w} = (w_1, \ldots, w_{n+1}) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $gatp(v'_1, \ldots, v'_n/F)$ and $gatp(w_1, \ldots, w_n/F)$ are $\downarrow^*$-independent extensions of $gatp(v_1, \ldots, v_n/E)$.
By induction hypothesis, $gatp(v'_1, \ldots, v'_n/F) = gatp(w_1, \ldots, w_n/F)$. Let U be an automorphism of the monster model fixing E pointwise such that for every $j = 1, \ldots, n$, $U(v_j) = v'_j$ and let U' an automorphism of the monster model fixing F pointwise such that for every $j = 1, \ldots, n$, $U'(v'_j) = w'_j$. Again by transitivity, $gatp(U^{-1}(v'_{n+1})/Fv_1 \cdots v_n)$ and $gatp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ are $\downarrow^*$-independent extensions of $gatp(v_{n+1}/Ev_1, \cdots v_n)$. By the case $n = 1$ $gatp(U^{-1}(v'_{n+1})/Fv_1 \cdots v_n) = gatp((U' \circ U)^{-1}(w_{n+1})/Fv_1, \cdots v_n)$ and therefore

$$p' = gatp(v'_1, \ldots, v'_n v'_{n+1}/F) = gatp(w_1, \ldots, w_n, w_{n+1}/F) = q.$$

□

Definition 5.5.6. Let $\mathcal{K}$ be an homogeneous MAEC with monster model $\mathcal{M}$. Let $B \subseteq A \subseteq M$ and let $a \in M$. The type $gatp(a/A)$ is said to *split* over B if there are b, $c \in A$ such that

$$gatp(b/B) = gatp(c/B)$$

but

$$gatp(b/Ba) \neq gatp(c/Ba)$$

Theorem 5.5.7. *Let $v \in \tilde{H}$ and let $F \subseteq G \subseteq \tilde{H}$. If $gatp(v/G)$ splits over F then $v \not\downarrow^*_F G$.*

Proof. If $gatp(v/G)$ splits over F, then there are two vectors w_1 and $w_2 \in G$ such that $gatp(w_1/F) = gatp(w_2/F)$ but $gatp(w_1/Fv) \neq gatp(w_2/Fv)$. Then, either

$$gatp(P^{\perp}_{ga-acl(Fv)}w_1/\emptyset) \neq gatp(P^{\perp}_{ga-acl(Fv)}w_2/\emptyset)$$

or

$$P_{ga-acl(Fv)}w_1 \neq P_{ga-acl(Fv)}w_2$$

Let us consider each case:

Case $gatp(P^{\perp}_{ga-acl(Fv)}w_1/\emptyset) \neq gatp(P^{\perp}_{ga-acl(Fv)}w_2/\emptyset)$ Since

$$P^{\perp}_{ga-acl(Fv)}w_1 = P^{\perp}_{ga-acl(F)}w_1 - P_{P^{\perp}_{ga-acl(F)}v_e}w_1$$

and

$$P^{\perp}_{ga-acl(Fv)}w_2 = P^{\perp}_{ga-acl(F)}w_2 - P_{P^{\perp}_{ga-acl(F)}v_e}w_2,$$

this means that

$$gatp(P_{P^{\perp}_{ga-acl(F)}v_e}w_1/\emptyset) \neq gatp(P_{P^{\perp}_{ga-acl(F)}v_e}w_2/\emptyset)$$

So, either $P_{P^{\perp}_{ga-acl(F)}v_e}w_1 \neq 0$ or $P_{P^{\perp}_{ga-acl(F)}v_e}w_2 \neq 0$. Let us suppose without loss of generality that $P_{P^{\perp}_{ga-acl(F)}v_e}w_1 \neq 0$. Then $P_{w_1}(P^{\perp}_{ga-acl(F)}v_e) \neq 0$, which implies that $P_{ga-acl(F)}v \neq P_{ga-acl(Fw_1)}v$. That is, $v \not\downarrow^{*}_{F} w_1$ and by transitivity, $v \not\downarrow^{*}_{F} G$.

Case $P_{ga-acl(Fv)}w_1 \neq P_{ga-acl(Fv)}w_2$ Since

$$P_{ga-acl(Fv)}w_1 = P_{ga-acl(F)}w_1 + P_{P^{\perp}_{ga-acl(F)}v_e}w_1$$

and

$$P_{ga-acl(Fv)}w_2 = P_{ga-acl(F)}w_2 + P_{P^{\perp}_{ga-acl(F)}v_e}w_2,$$

this means that $P_{P^{\perp}_{ga-acl(F)}v_e}w_1 \neq P_{P^{\perp}_{ga-acl(F)}v_e}w_2$ and, therefore either $P_{P^{\perp}_{ga-acl(F)}v_e}w_1 \neq 0$ or $P_{P^{\perp}_{ga-acl(F)}v_e}w_2 \neq 0$. As in previous item, this implies that $v \not\downarrow^{*}_{F} G$.

□

Theorem 5.5.8. *Let $v \in \tilde{H}$ and $F \subseteq G \subseteq \tilde{H}$ such that $F = ga - acl(F)$ and G is $|F|$-saturated. If $v \not\downarrow^{*}_{F} G$, then $gatp(v/G)$ splits over F.*

Proof. If $v \not\downarrow^{*}_{F} G$ then $w := P_G v - P_F v \neq 0$ and $w \perp F$. Since G is $|F|$-saturated, there is $w' \in G$ such that $gatp(w/F) = gatp(w'/F)$ and $w' \perp P_G v$. Since $\langle v \mid w \rangle \neq 0$, $P_v w \neq 0$, while $P_v w' = 0$. □

Fact 5.5.9 (Theorem 14.14 in [21])**.** A first order continuous logic theory T is stable if and only if there is an independence relation $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfying local character, finite character of dependence, transitivity, symmetry, invariance, existence and stationarity. In that case the relation $\mathop{\smile\!\!\!\!\!\mid}^*$ coincides with non-forking.

Remark 5.5.10. In the MAEC $\mathcal{K}_{(H,D,\pi)}$ splitting has the same properties as non-forking of a superstable theory first order theory.

Recall that a canonical base for a type p is a minimal set over which p is independent. In general, this smallest tuple is an imaginary, but in Hilbert spaces it corresponds to a tuple of real elements. Next theorem gives an explicit description of canonical bases for types in the structure, again we get a tuple of real elements.

Theorem 5.5.11. *Let* $\bar{v} = (v_1, \dots, v_n) \in H^n$ *and* $E \subseteq H$. *Then* $Cb(gatp(\bar{v}/E)) := \{(P_E v_1, \dots, P_E v_n)\}$ *is a canonical base for the type* $gatp(\bar{v}/E)$.

Proof. First of all, we consider the case of a 1-tuple. By Theorem 5.5.3 $gatp(v/E)$ does not fork over $Cb(gatp(v/E))$. Let $(v_k)_{k<\omega}$ a Morley sequence for $gatp(v/E)$. We have to show that $P_E v \in dcl((v_k)_{k<\omega})$. By Theorem 5.5.3, for every $k < \omega$ there is a vector w_k such that $v_k = P_E v + w_k$ and $w_k \perp ga-acl(\{P_E v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_e$ and for all j, $k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v'_k := \frac{v_1+\cdots+v_k}{n} = P_E v + \frac{w_1+\cdots+w_k}{n}$. Then for every $k < \omega$, $v'_k \in dcl((v_k)_{k<\omega})$. Since $v'_k \to P_E v$ when $k \to \infty$, we have that $P_E v \in dcl((v_k)_{k<\omega})$.
For the case of a general n-tuple, by Remark 5.5.2, it is enough to repeat the previous argument in every component of $\bar{v}$. □

Recall that a theory is said to be uniformly finitely based if for every $\epsilon > 0$ $\exists N \in \mathbb{N}$ such that if $(v_k)_{k<\omega}$ is a Morley sequence for $gatp(v/E)$ then $P_E(v) \in dcl_\epsilon(v_1, \dots, v_N)$, where $dcl_\epsilon(v_1, \dots, v_N)$ is the set of vectors with distance to $dcl(v_1, \dots, v_N)$ less than ϵ.

Corollary 5.5.12. $\mathcal{K}_{(H,D,\pi)}$ *is uniformly finitely based.*

Proof. Clear by previous theorem. □

5.6 Orthogonality and domination

In this section, we characterize orthogonality and domination of types in terms of orthogonality and domination of their associated functionals on the double commutant of the monster model of the class.

Definition 5.6.1. Let $\mathcal{A}'$ be the dual space of $\mathcal{A}$. An element $\phi \in \mathcal{A}'$ is called *positive* if $\phi(a) \geq 0$ whenever $a \in \mathcal{A}$ is positive, i.e. there is $b \in \mathcal{A}$ such that $a = b^*b$. The set of positive functionals is denoted by $\mathcal{A}'_+$.

Lemma 5.6.2. *Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H, and let $v \in H$. Then the function ϕ_v on $\mathcal{A}$ such that for every $S \in \mathcal{A}$, $\phi_v(S) = \langle Sv \mid v\rangle$ is a positive linear functional.*

Definition 5.6.3. Let ϕ and ψ be positive linear functionals on $\mathcal{A}$.

1. They are called *orthogonal* ($\phi \perp \psi$) if $\|\phi - \psi\| = \|\phi\| + \|\psi\|$.
2. Also, ϕ is called *dominated* by ψ ($\phi \leq \psi$) if there exist $\gamma > 0$ such that the functional $\gamma\psi - \phi$ is positive.

Theorem 5.6.4. *Let $v, w \in H$. Then (H_v, D_v, π_v) is isometrically isomorphic to a subrepresentation of (H_w, D_w, π_w) if and only if $\phi_v \leq \phi_w$, where ϕ_v and ϕ_w are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_v(S) = \langle Sv|v\rangle$ and $\phi_w(S) = \langle Sw|w\rangle$.*

Proof. Suppose (H_v, D_v, π_v) is isometrically isomorphic to a subrepresentation of (H_w, D_w, π_w). Then there exists a vector $v' \in H_w$ such that $(H_v, D_v, \pi_v) \simeq (H_{v'}, D_{v'}, \pi_{v'})$. By Radon Nikodim Theorem for rings of operators (see [16]), there exists a bounded positive operator $P : (H_w, D_w, \pi_w) \to (H_{v'}, D_{v'}, \pi_{v'})$ such that $Pw = v'$ and P commutes with every element of $\pi_v(\mathcal{A})''_w$. Let $\gamma = \|P\|^2$. Then, for every positive element $a \in \mathcal{A}$, $\phi_v(a) = \phi_{v'}(a) = \langle \pi(a)v' \mid v'\rangle = \langle \pi(a)Pw \mid Pw\rangle = \langle P^*\pi(a)Pw \mid w\rangle = \langle \pi(a)\|P\|^2 w \mid w\rangle \leq \gamma\langle \pi(a)w \mid w\rangle = \gamma\phi_w(a)$ which means that $\gamma\phi_w - \phi_v$ is positive and $\phi_v \leq \phi_w$.
The converse comes from Gelfand Naimark Segal Construction .5.32 ☐

Lemma 5.6.5. *Let $v, w \in H$. If $\phi_v \perp \phi_w$, then $(H_v, , D_v, \pi_v)$ is not isometrically isomorphic to any subrepresentation of (H_w, D_w, π_w).*

Proof. Suppose $\phi_v \perp \phi_w$, and (H_v, D_v, π_v) is isometrically isomorphic to subrepresentation of (H_w, D_w, π_w). By Theorem 5.6.4 $\phi_v \leq \phi_w$; let $\gamma > 0$ be a real number such that $\gamma\phi_w - \phi_v$ is a bounded positive functional on $\pi(\mathcal{A})''$ and let $u \in H$ be such that $\phi_u = \gamma\phi_w - \phi_v$, which is possible by GNS Theorem. Then $\phi_v = \gamma\phi_w - \phi_u$, and $\|\phi_w - \phi_v\| = \|\phi_w - \gamma\phi_w + \phi_u\| = \|(1-\gamma)\phi_w + \phi_u\| = |1-\gamma|\|\phi_w\| + \|\phi_u\| \neq \|\phi_w\| + \|\phi_v\|$, but this contradicts $\phi_v \perp \phi_w$. ☐

Here, a few facts that will be needed to prove Theorem 5.6.12:

Remark 5.6.6. Recall that two representations are said to be *disjoint* if they do not have any common subrepresentation up to isometric isomorphism.

Fact 5.6.7 (Proposition 3 in [12], Chapter 5, Section 2)**.** Two subrepresentations (H_1, D_1, π_1), (H_2, D_2, π_2) of (H, D, π) are disjoint if and only if there is a projection P in $\pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that if P_1 and P_2 are the projections on H_1 and H_2 respectively, we have that $PP_1 = P_1$ and $(I - P)P_2 = P_2$.

Fact 5.6.8 (Corollary 2.2.5 in [31])**.** Let $\mathcal{A}$ be a C^*algebra and $\pi : \mathcal{A} \to B(H)$ a nondegenerate representation of $\mathcal{A}$. Let $\mathcal{M}$ be the strong closure of $\pi(\mathcal{A})$. Then $\mathcal{M}$ is weakly closed and $\mathcal{M} = \mathcal{A}''$.

Theorem 5.6.9 (Kaplanski densitiy theorem. Theorem 2.3.3. in [31])**.** *Let $\mathcal{A}$ be a C^*-subalgebra of $B(H)$ with strong closure $\mathcal{M}$. Then the unit ball $Ball_1(\mathcal{A})$ of $\mathcal{A}$ is strongly dense in the unit ball $Ball_1(\mathcal{M})$ of $\mathcal{M}$. Furthermore, the set of selfadjoint elements in $Ball_1(\mathcal{A})$ is strongly dense in the set of selfadjoint elements of $Ball_1(\mathcal{M})$.*

Fact 5.6.10 (Lemma 3.2.3 in [31])**.** Let ϕ and ψ be two positive linear functionals on $\pi(\mathcal{A})_w^{\perp\perp}$. Then, $\phi \perp \psi$ if and only if for all $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi(e-a) < \epsilon$ and $\psi(a) < \epsilon$.

Lemma 5.6.11. *Let ϕ_1, ϕ_2, ψ_1 and ψ_2 be positive linear functionals on $\pi(\mathcal{A})_w^{\perp\perp}$ such that $\phi_1 \leq \phi_2$ and $\psi_1 \leq \psi_2$. If $\phi_2 \perp \psi_2$, then $\phi_1 \perp \psi_1$.*

Proof. Let $\gamma_1 > 0$ and $\gamma_2 > 0$ be such that $\gamma_1\phi_2 - \phi_1$ and $\gamma_2\psi_2 - \psi_1$ are positive. By Fact 5.6.10, for $\epsilon > 0$ there exists a positive $a \in \mathcal{A}$ with norm less than or equal to 1 such that $\phi_2(e-a) < \frac{\epsilon}{\gamma_1+\gamma_2}$ and $\psi_2(a) < \frac{\epsilon}{\gamma_1+\gamma_2}$. Then $\phi_1(e-a) \leq \gamma_1\phi_2(e-a) < \frac{\gamma_1\epsilon}{\gamma_1+\gamma_2} < \epsilon$ and $\psi_1(a) \leq \gamma_2\psi_2(a) < \frac{\gamma_2\epsilon}{\gamma_1+\gamma_2} < \epsilon$. □

Theorem 5.6.12. *Let $v, w \in H$. $\phi_v \perp \phi_w$ if and only if no subrepresentation of $(H_v, D_v\pi_v)$ is isometrically isomorphic to a subrepresentation of (H_w, D_w, π_w).*

Proof. Suppose $\phi_v \perp \phi_w$. By Lemma 5.6.11, if $(H_{v'}, D_{v'}, \pi_{v'})$ is a subrepresentation of (H_v, D_v, π_v) and $(H_{w'}, D_{w'}, \pi_{w'})$ is a subrepresentation of (H_w, D_w, π_w), then $\phi_{v'} \perp \phi_{w'}$, by Lemma 5.6.5, $(H_{v'}, D_{v'}, \pi_{v'})$ is not isometrically isomorphic to $(H_{w'}, D_{w'}, \pi_{w'})$, and the conclusion follows.
Conversely, suppose no subrepresentation of (H_v, D_v, π_v) is isometrically isomorphic to a subrepresentation of (H_w, D_w, π_w). Then the representations (H_v, D_v, π_v) and (H_w, D_w, π_w) are disjoint. By Fact 5.6.7, there is a projection $P \in \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that $PP_v = P_v$ and $(I-P)P_w = P_w$. Then, $\phi_v(I-P) = \langle (I-P)v \mid v\rangle = \langle (v - PP_v v) \mid v\rangle = \langle (v-v) \mid v\rangle = 0$. On the other hand, $\phi_w(P) = \langle Pw \mid w\rangle = \langle w - (w - Pw) \mid w\rangle = \langle w - (I-P)w \mid w\rangle = \langle w - (I-P)P_w w \mid w\rangle = \langle w - P_w w \mid w\rangle = \langle w - w \mid w\rangle = 0$. By Fact 5.6.8 and Theorem 5.6.9, the projection P is strongly approximable by positive elements in $\pi(\mathcal{A})$ and therefore, for $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi_v(e-a) < \epsilon$ and $\phi_w(a) < \epsilon$. By Fact 5.6.10, $\phi_v \perp \phi_w$. □

Definition 5.6.13. Let $A \subseteq H$ and $p, q \in S_n(A)$. We say that p is *almost orthogonal* to q ($p \perp^a q$) if for all $\bar{a} \models p$ and $\bar{b} \models q$ $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$.

Definition 5.6.14. Let $A \subseteq \mathfrak{B}$ and $p \in S_n(A)$ and $q \in S_n(B)$ two stationary types. We say that p is *orthogonal* to q ($p \perp q$) if for all $B \supseteq A \cup B$, $p_B \supseteq p$ non-forking extension, and $q_B \supseteq q$ non-forking extension, $p_B \perp^w q_B$

Lemma 5.6.15. *Let $p, q \in S_1(\emptyset)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\phi_{v_e} \perp \phi_{w_e}$, where ϕ_{v_e} and ϕ_{w_e} are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_{v_e}(S) = \langle Sv_e | v_e\rangle$ and $\phi_{w_e}(S) = \langle Sw_e | w_e\rangle$.*

Proof. Suppose $p \perp^a q$. By Remark 5.5.2, this implies that $H_{v_e} \perp H_{w_e}$ for all $v \models p$ and $w \models q$. Let $v \models p$ and $w \models q$. Then no subrepresentation of $(H_{v_e}, D_{v_e}, \pi_{v_e})$ is isometrically isomorphic to any subrepresentation of $(H_{w_e}, D_{w_E}, \pi_{w_e})$. By Lemma 5.6.12, this implies that $\phi_{v_e} \perp \phi_{w_e}$.
Conversely, if $p \not\perp^a q$ there are v, $w \in H$ such that $v \models p$, $w \models q$ and $H_{v_e} \not\perp H_{w_e}$. This implies that there exist elements a_1, $a_2 \in \mathcal{A}$ such that $\pi(a_1)v_e \not\perp \pi(a_2)w_e$. This means that $0 \neq \langle \pi(a_1)v_e \mid \pi(a_2)w_e\rangle = \langle v_e \mid \pi(a_1^* a_2)w_e\rangle$. So, we can assume that there exists an element $a \in \mathcal{A}$ such that $v_e \not\perp \pi(a)w_e$. Since $v_e = P_{w_e}v_e + P^{\perp}_{w_e}v_e$ and $P_{w_e}v_e \neq 0$, we can prove that $\phi_{P_{w_e}v_e} \leq \phi_{v_e}$ by using a procedure similar to the one used in the proof of Theorem 5.6.4 and, since $P_{w_e}v_e \in H_{w_e}$, we get $\phi_{P_{w_e}v_e} \leq \phi_{w_e}$. By Lemma 5.6.11, this implies that $\phi_{v_e} \not\perp \phi_{w_e}$. □

Theorem 5.6.16. *Let $E \subseteq H$. Let $p, q \in S_1(E)$, let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \perp^a_E q$ if and only if $\phi_{P^{\perp}_E(v_e)} \perp \phi_{P^{\perp}_E(w_e)}$, where $\phi_{P^{\perp}_E(v_e)}$ and $\phi_{P^{\perp}_E(w_e)}$ are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_{P^{\perp}_E(v_e)}(S) = \langle SP^{\perp}_E(v_e)|P^{\perp}_E(v_e)\rangle$ and $\phi_{P^{\perp}_E(w_e)}(S) = \langle SP^{\perp}_E(w_e)|P^{\perp}_E(w_e)\rangle$.*

Proof. Clear by Lemma 5.6.15. □

Theorem 5.6.17. *Let $E \subseteq H$. Let $p, q \in S_1(E)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Assume $p \perp^a q$, $E \subseteq F \subseteq H$ are small subsets of the monster model and p', $q' \in S_1(F)$ are non-forking extensions of p and q respectively. Let v, $w \in H$ be such that $v \models p'$ and $w \models q'$, then $\phi_{P^{\perp}_F(v_e)} = \phi_{P^{\perp}_E v_e} \perp \phi_{P^{\perp}_E w_e} = \phi_{P^{\perp}_F(w_e)}$. By Lemma 5.6.15, this implies that $p' \perp^a q'$. Therefore $p \perp q$.
The converse is trivial. □

Definition 5.6.18. Let A, $\mathfrak{B}$ be small subsets of $\tilde{H}$ and $p \in S_n(A)$ and $q \in S_n(B)$ two stationary types. We say that p *dominates* q over a set $C \supseteq A \cup B$ ($p \rhd_C q$) if there exist v $w \in \tilde{\mathcal{H}}$ such that $gatp(v/C)$ is a non-forking extension of p, $gatp(w/C)$ is a non-forking extension of q and for all $D \supseteq C$ if $v \downarrow^*_C D$ then $w \downarrow^*_C D$

Lemma 5.6.19. *Let $p, q \in S_1(\emptyset)$ and let v, $w \in H$ be such that $v \models p$ and $w \models q$. Then, $p \rhd_\emptyset q$ if and only if $\phi_{w_e} \leq \phi_{v_e}$, where ϕ_{v_e} and ϕ_{w_e} are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_{v_e}(S) = \langle Sv_e|v_e\rangle$ and $\phi_{w_e}(S) = \langle Sw_e|w_e\rangle$.*

Proof. Suppose $p \rhd_\emptyset q$. Suppose that v' and w' are such that $v' \models p$, $w' \models q$ and if $v' \downarrow^*_\emptyset E$ then $w' \downarrow^*_\emptyset E$ for every E. Then for every $E \subseteq H$

$$P_E v'_e = 0 \Rightarrow P_E w'_e = 0$$

This implies that $w'_e \in H_{v'_e}$, and $H_{w'_e} \subseteq H_{v'_e}$. By Theorem 5.6.4, $\phi_{w_e} = \phi_{w'_e} \leq \phi_{v'_e} = \phi_{v_e}$.
For the converse, suppose $\phi_{w_e} \leq \phi_{v_e}$. Then, by Theorem 5.6.4 H_{w_e} is isometrically isomorphic to a subrepresentation of H_{v_e}, which implies that there is $w' \in H_v$ such that $w' \models gatp(w/\emptyset)$ and for every $E \subseteq H$

$$P_E v_e = 0 \Rightarrow P_E w'_e = 0$$

This means than $gatp(w/\emptyset) \lhd_\emptyset gatp(v/\emptyset)$. □

Theorem 5.6.20. *Let E, F and G be small subsets of $\tilde{H}$ such that $E, F \subseteq G$ and $p \in S_1(E)$ and $q \in S_1(F)$ be two stationary types. Then $p \rhd_G q$ if and only if there exist $v, w \in \tilde{H}$ such that $gatp(v/G)$ is a non-forking extension of p, $gatp(w/G)$ is a non-forking extension of q and $\phi_{P^\perp_{ga-acl(G)} w_e} \leq \phi_{P^\perp_{ga-acl(G)} v_e}$, where $\phi_{P^\perp_E(v_e)}$ and $\phi_{P^\perp_E(w_e)}$ are the functionals defined on $\pi(\mathcal{A})''_w$ such that for every $S \in \pi(\mathcal{A})'_w$, $\phi_{P^\perp_E(v_e)}(S) = \langle SP^\perp_E(v_e) | P^\perp_E(v_e)\rangle$ and $\phi_{P^\perp_E(w_e)}(S) = \langle SP^\perp_E(w_e) | P^\perp_E(w_e)\rangle$.*

Proof. Clear by Lemma 5.6.19. □

.1 Continuous Logic and Model Theory for Metric Structures

Continuous Logic begins in the 1960's with the work of Chang and Keisler: *Continuous Model Theory* (see[11]). There, the authors developed the concepts of continuous logic such as, elementary equivalence, ultraproducts, etc. Also, they proved a Lowenheim-Skölem theorem, among others.
Later, in the 1970's, C. Ward Henson developed a logic for Banach spaces where he proved an analytic version of Keisler-Shelah theorem: Two structures are elementarily equivalent if and only if they have some isomorphic ultrapowers.
In 1980's and middle 1990's, Iovino and Henson, based on previous work of Krivine, adapted tools of stability theory to Banach space theory. These ideas were generalized by Ben Yaacov with the notion of *Compact Abstract Theory* (CAT) which includes both first order logic and Henson's logic for Banach space structures.
Finallly, in 2005, Ben Yaacov, Berenstein, Henson y Usvyatsov introduced continuous logic for metric structures, which is suitable for the study of model theory of metric spaces.
The main source for this section is [21]. We fix a bounded complete metric space (M, d) of diameter 1.

Definition .1.1. A *predicate* on M is an uniformly continuous function from M^n into the the interval $[0, 1]$.

Definition .1.2. A *function* on M is an uniformly continuous function from M^n into M.

Definition .1.3. Let $f : X \to Y$ be a uniformly continuous function from a metric space X to another metric space Y. A *modulus of uniform continuity* for f is a function $\Delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all x, y and for all ϵ:

$d(x, y) < \Delta(\epsilon) \implies d(f(x), f(y)) \leq \epsilon.$

Definition .1.4. A *metric structure* based on a bounded complete metric space (M, d) consists of the metric space M, an indexed family $(R_i|i \in I)$ of predicates on M, an indexed family $(F_j|j \in J)$ of functions on powers of M, and an indexed family $(a_k|k \in K)$ of distinguished elements of M. We write this structure as

$\mathcal{M} = (M, d, R_i, F_j, a_k|i \in I, j \in J, k \in K).$

When we study a Banach space X, we usually consider the space as a many sorted structure, where there is a sort for each closed ball of positive integer radius around the origin. For instance, the unit ball of X is a space of diameter 2.

Definition .1.5. For a metric structure $\mathcal{M}$, its *signature* $L(\mathcal{M})$ is the union of the indexed families of predicates, functions and constants along with natural numbers corresponding to the arity, and functions corresponding to the modulus of uniform continuity for every symbol.

Definition .1.6. For a signature L, the *L-terms* are constructed inductively as follows:

1. Each variable and constant symbol is an L-term.

2. If F is an n-ary function and $t_1, \ldots, t_n$ are L-terms, then $F(t_1, \ldots, t_n)$ is an L-term.

Definition .1.7. For a signature L, the *atomic L-formulas* are expresions of the form $d(t_1, t_2)$ or $R(t_1, \ldots, t_n)$ for R a predicate symbol in L and $t_, \ldots, t_n$ terms.

Definition .1.8. For a signature L, the *L-formulas* are constructed inductively as follows:

1. Atomic L-formulas are L-formulas.

2. If $u : [0, 1]^n \to [0, 1]$ is continuous and $\phi_1, \ldots, \phi_n$ are L-formulas, then $u(\phi_1, \ldots, \phi_n)$ is an L-formula.

3. If ϕ is an L-formula and x is a variable, then $\sup_x \phi$ and $\inf_x \phi$ are L-formulas.

Remark .1.9. $\sup_x$ and $\inf_x$ play the role of quantifiers in first order logic.

Remark .1.10. A quntifier free formula is a formula constructed using only the rules $(1), (2)$.

Remark .1.11. The notions of subformula, free and bounded variables can be naturally generalized to continuous logic in the natural way.

Definition .1.12. An *L-sentence* is an L-formula which has no free variables.

Remark .1.13. When t is a term and the variables occurring in it are among the variables $x_1, \ldots, x_n$, we indicate this by writing t as $t(x_1, \ldots, x_n)$.

Remark .1.14. Given an L-term $t(x_1, \ldots, x_n)$, the *interpretation of* t in $\mathcal{M}$, is a function $t^{\mathcal{M}} : M^n \to M$ defined inductively as in first order logic.

Definition .1.15. We define inductively the *value* of an L-formula $\sigma(a_1, \cdots, a_n)$ with $a_1, \cdots, a_n \in \mathcal{M}$, denoted by $\sigma^{\mathcal{M}}$, as follows:

1. $(d(t_1(a_1, \cdots, a_n), t_2(a_1, \cdots, a_n)))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \cdots, a_n), t_2^{\mathcal{M}}(a_1, \cdots, a_n))$ for any L-terms t_1, t_2.

2. $(P(t_1(a_1, \cdots, a_n), \ldots, t_2(a_1, \cdots, a_n)))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \cdots, a_n), \ldots, t_n^{\mathcal{M}}(a_1, \cdots, a_n))$ for any predicate symbol $P \in L$ and any L-terms $t_1(a_1, \cdots, a_n), \ldots, t_n(a_1, \cdots, a_n)$.

3. $(u(\sigma_1, \ldots, \sigma_n))^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \ldots, \sigma_n^{\mathcal{M}})$ for any continuous $u : [0, 1]^n \to [0, 1]$ and any L-sentences $\sigma_1, \ldots, \sigma_n$.

4. $(\sup_x \phi(a_1, \cdots, a_n, x))^{\mathcal{M}}$ is the supremum in $[0, 1]$ of the set $\{\phi(a_1, \cdots, a_n, a)^{\mathcal{M}} | a \in M\}$ for any L-formula $\phi(x)$.

5. $(\inf_x \phi(a_1, \cdots, a_n, x))^{\mathcal{M}}$ is the infimum in $[0,1]$ of the set $\{\phi(a_1, \cdots, a_n, a)^{\mathcal{M}} | a \in M\}$ for any L-formula $\phi(x)$.

Remark .1.16. $\sup \emptyset = 0$ and $\inf \emptyset = 1$

Definition .1.17. An *L-statement* E is an expression of the form $\phi = \psi$ or $\phi \leq \psi$, where ϕ and ψ are L-formulas. We call E *closed* if both ϕ and ψ are sentences. Similarly if ϕ and ψ are quantifier free formulas E is called a *quantifier free* statement.

Definition .1.18. If E is the L-statement $\phi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we say that E *is true of* $a_1, \ldots, a_n \in M$ and write $\mathcal{M} \models E(a_1, \ldots, a_n)$ if

$$\phi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{M}}(a_1, \ldots, a_n).$$

Similarly, if E is the L-statement $\phi(x_1, \ldots, x_n) \leq \psi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we say that E *is true of* $a_1, \ldots, a_n \in M$ if

$$\phi^{\mathcal{M}}(a_1, \ldots, a_n) \leq \psi^{\mathcal{M}}(a_1, \ldots, a_n).$$

Definition .1.19. Two L-formulas ϕ and ψ are said to be *logically equivalent* (or just equivalent) if the L-statement $\phi = \psi$ is true in every L-structure.

Definition .1.20. For $i = 1, 2$, let E_i be the statement $\phi_i = \psi_i$. We say that E_1 and E_2 are logically equivalent if for every L-structure $\mathcal{M}$ and every $a_1, \ldots, a_n$ we have:

$$\mathcal{M} \models E_1(a_1, \ldots, a_n) \text{ iff } \mathcal{M} \models E_2(a_1, \ldots, a_n).$$

Definition .1.21. We define a binary function $\dot{-} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by:

$$x \dot{-} y = \begin{cases} (x - y) & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

Remark .1.22. If $x, y \in [0,1]$, then $x \dot{-} y \in [0,1]$.

Remark .1.23. Every statement is equivalent to one of the form $\phi = 0$. For example, the statement $\phi \leq \psi$ is equivalent to the statement $|\phi \dot{-} \psi| = 0$.

Definition .1.24. Fix a signature L. An *L-theory* is a set of closed L-statements. If T is a theory in L and $\mathcal{M}$ is an L-structure, we say that $\mathcal{M}$ is a model of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models E$ for every statement $E \in T$. We write $Mod_L(T)$ for the collection of all L-structures that are models of T. If $\mathcal{M}$ is an L-structure, the *theory* of $\mathcal{M}$, denoted $Th(\mathcal{M})$, is the set of closed L-statements which are true in $\mathcal{M}$. A theory of this form is called *complete*.

Definition .1.25. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures.

1. $\mathcal{M}$ and $\mathcal{N}$ are said to be *elementary equivalent*, and we denote it by $\mathcal{M} \equiv \mathcal{N}$, if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ. Equivalently, this holds if $Th(\mathcal{M}) = Th(\mathcal{N})$.

2. If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, and write $\mathcal{M} \prec \mathcal{N}$, if for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \mathcal{M}$, $\phi^{\mathcal{M}}(a_1, \ldots, a_n) = \phi^{\mathcal{N}}(a_1, \ldots, a_n)$. In this case we say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

3. A function F from a subset of M into N is a *partial elementary map* from $\mathcal{M}$ into $\mathcal{N}$ if for all L-formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n$ in the domain of F, we have:

   $\phi^{\mathcal{M}}(a_1, \ldots, a_n) = \phi^{\mathcal{N}}(F(a_1), \ldots, F(a_n))$

4. An *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an elementary map from $\mathcal{M}$ into $\mathcal{N}$ whose domain is all $\mathcal{M}$.

Fact .1.26 (Tarski-Vaught Test, proposition 4.5 of [21])**.** Let $\mathcal{M}$, $\mathcal{N}$ be L-structures with $\mathcal{M} \subseteq \mathcal{N}$. The following are equivalent:

1. $\mathcal{M} \prec \mathcal{N}$.

2. for every L-formula $\phi(x_1, \ldots, x_n)$ and every $\epsilon > 0$ the following condition holds: If $a_1, \ldots, a_n \in M$ and $b \in N$, there exists $c \in M$ such that:

   $|\phi^{\mathcal{N}}(a_1, \ldots, a_n, c) - \phi^{\mathcal{N}}(a_1, \ldots, a_n, b)| \leq \epsilon.$

Definition .1.27. Suppose that $\mathcal{M}$ is an L-structure with underlying metric space (M, d) and $A \subseteq M$. Denote the $L(A)$-structure $(\mathcal{M}, a)_{a \in A}$ by $\mathcal{M}_A$. Let $e_1, \ldots, e_n \in M$ and fix distinct variables $x_1, \ldots, x_n$. Then

- The *type* of $(e_1, \ldots, e_n)$ *over* A *in* $\mathcal{M}$, denoted by $tp_{\mathcal{M}}(e_1, \ldots, e_n/A)$, is the set of $L(A)$-statements $E(x_1, \ldots, x_n)$ such that

  $\mathcal{M}_A \models E(e_1, \ldots, e_n)$

- The *quantifier free type* of $(e_1, \ldots, e_n)$ *over* A *in* $\mathcal{M}$, denoted by $qftp_{\mathcal{M}}(e_1, \ldots, e_n/A)$, is the set of quantifier free $L(A)$-statements $E(x_1, \ldots, x_n)$ such that

  $\mathcal{M}_A \models E(e_1, \ldots, e_n)$

  If the structure $\mathcal{M}$ is clear from the context, we omit it and write $tp(e_1, \ldots, e_n/A)$ or $qftp(e_1, \ldots, e_n/A)$ respectively.

Remark .1.28. 1. Let $\mathcal{M}$ and A as above, and let $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ be elements of $\mathcal{M}$. Then

   $tp_{\mathcal{M}}(e_1, \ldots, e_n/A) = tp_{\mathcal{M}}(e'_1, \ldots, e'_n/A)$

   if and only if

   $(\mathcal{M}_A, e_1, \ldots, e_n) \equiv (\mathcal{M}_A, e'_1, \ldots, e'_n)$

2. If $(\mathcal{M}, A)$ are as above and $\mathcal{M} \prec \mathcal{N}$, then

   $tp_{\mathcal{M}}(e_1, \dots, e_n/A) = tp_{\mathcal{N}}(e_1, \dots, e_n/A)$.

Let L be a signature for metric structures, and let $L(A)$ be the extension of L with a set A of new constant symbols. Let T_A denote a complete theory in $L(A)$ and let T be the restriction of T_A to L.

Definition .1.29. A set p of $L(A)$-statements whose free variables are among $x_1, \dots, x_n$ is called an *n-type* over A if there exists a model $(\mathcal{M}, a)_{a\in A}$ of T_A and elements $e_1, \dots, e_n$ of M such that $p(x_1, \dots, x_n) = tp_{\mathcal{M}}(e_1 \dots, e_n/A)$. In this case, we say that $(e_1, \dots, e_n)$ *realizes* p.

Definition .1.30. The collection of all such n-types over A is denoted by $S_n(T_A)$, or just $S_n(A)$ if T_A is clear from the context. If $A = \emptyset$ we write $S_n(T)$ instead of $S_n(T_\emptyset)$.

Fact .1.31 (Remark 8.3 of [21])**.** There is a model $(\mathcal{M}, a)_{a\in A}$ of T_A such that for all $n \geq 1$, every n-type over A is realized.

Definition .1.32. Let L be a signature and μ an infinite cardinal. A L-structure $\mathcal{M}$ is called *μ-saturated* if every type $p \in S_n^{\mathcal{M}}(A)$ over a subset $A \subseteq M$ with $|A| < \mu$ is realized in $\mathcal{M}$.

Definition .1.33. The *monster model* for a theory T is a μ-saturated and μ-strongly homogeneous structure for a cardinal μ greater than the size of any set of parameters under consideration, and is denoted by $\tilde{\mathcal{M}}$.

Definition .1.34. Let $\phi(x_1, \dots, x_n)$ be an $L(A)$-formula and $\epsilon > 0$. Then

$(\phi, \epsilon) = \{q \in S_n(T_A)|$ the statement $\phi \leq \delta$ belongs to q for some $\delta < \epsilon\}$.

The *logic topology* on $S_n(T_A)$ is defined as follows: If $p \in S_n(T_A)$ the basic neighborhoods of p are the sets of the form (ϕ, ϵ) where $\phi = 0$ is in p and $\epsilon > 0$.

Definition .1.35. Let $\mathcal{M}_A$ be any model of T_A in which every type in $S_{2n}(T_A)$ is realized, for $n \geq 1$. Let (M, d) the underlying metric space of $\mathcal{M}$ and let $p, q \in S_n(T_A)$. Then,

$d(p, q) = \inf\{\max_j d(a_j, b_j)|\mathcal{M} \models p(a_1, \dots, a_n)$ and $\mathcal{M} \models q(b_1, \dots, b_n)\}$

Remark .1.36. Note that $(S_n(T_A), d)$ is a metric space.

Definition .1.37. Let $p \in S_n(T)$ and $\mathcal{M} \models T$. Then $p(\mathcal{M}) = \{(a_1, \dots, a_n) \in M^n|\mathcal{M} \models p(a_1 \dots, a_n)\}$.

Definition .1.38. Let T be an L-theory and $\phi(x_1, \dots, x_n)$ be an L-formnula. Then ϕ is *approximable in T by quantifier free formulas* if for every $\epsilon > 0$ there is a quantifier-free L-formula $\psi(x_1, \dots, x_n)$ such that for all $\mathcal{M} \models T$ and all $a_1, \dots, a_n \in M$:

$|\phi^{\mathcal{M}}(a_1, \dots, a_n) - \psi^{\mathcal{M}}(a_1, \dots, a_n)| \leq \epsilon$

Definition .1.39. An L-theory T *admits quantifier elimination* if every L-formula is approximable in T by quantifier-free formulas.

closed

Definition .1.40. Let $A \subseteq \mathcal{M}$. A predicate $P : M^n \to [0,1]$ is A-*definable* in $\mathcal{M}$ if if there is a sequence $(\phi_n(\bar{x})|n \geq 1)$ of $L(A)$-formulas such that the predicates $\phi_n^{\mathcal{M}}(\bar{x})$ converge to $P(\bar{x})$ uniformly on M^n. A closed set $B \subseteq \mathcal{M}$ is A-*definable* if the distance $d(x, B)$ is A-definable in $\mathcal{H}$. A function is *definable* if its graph is definable.

Fact .1.41 (Proposition 9.19 of [21])**.** Let $D \subseteq \mathcal{M}$ be closed. Then D is definable in $\mathcal{M}$ if and only if there is a definable predicate $P : M^n \to [0,1]$ such that $P(x) = 0$ for all $x \in D$ and

$\forall\epsilon\exists\delta\forall x \in M^n(P(x) \leq \delta \Rightarrow d(x, D) \leq \epsilon)$.

Definition .1.42. Let T a theory, $p \in S_n(T)$ and $\mathcal{M}$ a complete model of T. Then p is *principal* if $p(\mathcal{M})$ is a definable subset of M.

Fact .1.43 (Omitting types Theorem, local version, Theorem 12.6 of [21])**.** Let T a complete theory in a countable signature, and let $p \in S_n(T)$. The following conditions are equivalent:

1. p is principal.

2. p is realized in every complete model of T.

Fact .1.44 (Theorem 12.10 of [21])**.** Let T a complete theory in countable signature. The following conditions are equivalent:

1. T is ω-categorical.

2. For each $n \geq 1$, every type $S_n(T)$ is principal.

3. For each $n \geq 1$, the metric space $(S_n(T), d)$ is compact.

Definition .1.45. Let $A \subseteq M$:

1. The *definable closure* of A in $\mathcal{M}$, denoted by $dcl_{\mathcal{M}}(A)$ (or just $dcl(A)$), is the set of all $a \in \mathcal{M}$ such that $\{a\}$ is A-definable in $\mathcal{M}$.

2. The *algebraic closure* of A in $\mathcal{M}$, denoted by $acl_{\mathcal{M}}(A)$ (or just $acl(A)$), is the union of all compact subsets of $\mathcal{M}$ that are A-definable.

3. If $\mathcal{M}$ is complete, the *bounded closure* of A in $\mathcal{M}$, denoted by $bdd_{\mathcal{M}}(A)$ (or just $bdd(A)$), is the set of all $a \in \mathcal{M}$ for which there is some cardinal μ such that for any $\mathcal{N} \succ \mathcal{M}$, the set of realizations of $tp(a/A)$ in $\mathcal{N}$ has cardinality less or equal to μ.

Fact .1.46 (Lemma 1.3 in [17])**.** Let L a signature, let $\mathcal{M}$ a complete L-structure, and $A \subseteq M$. Then, $acl_{\mathcal{M}}(A) = bdd_{\mathcal{M}}(A)$

.2 C^*-algebras, von Neumann algebras and factors

In this section we include the preliminaries about C^* and von Neumann algebras that are needed to understand the results in Chapter 2, mainly the notions of compact operator, factor and the type in which a factor can be classified. The main source for this section is [31].

Definition .2.1. Let $\mathcal{A}$ be a complex Banach algebra. $\mathcal{A}$ is called a C^*-algebra if there exists a map $^* : \mathcal{A} \to \mathcal{A}$, called *involution* such that for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

1. $(a+b)^* = a^* + b^*$
2. $(ab)^* = b^* a^*$
3. $(\alpha a)^* = \bar{\alpha} a^*$
4. $(a^*)^* = a$
5. $|a^* a| = |a|^2$

Fact .2.2. $\mathbb{C}$ is a C^*-algebra under complex conjugation.

Definition .2.3.
Let H be a Hilbert space

- Let H be a Hilbert space with norm denoted by $\|\cdot\|$. Let S be a linear operator from H into H. The operator S is called *bounded* if the set $\{\|Su\| : u \in H, \|u\| = 1\}$ is bounded in $\mathbb{C}$. If A is bounded we define the *norm* of A by:

  $$\|S\| = \sup_{u \in H, \|u\|=1} \|Su\|$$

- We denote by $B(H)$ the algebra of all bounded linear operators from H to H.
- Given a linear operator $S : H \to H$, its *adjoint operator*, denoted S^* is the unique linear operator $S^* : H \to H$ such that for every $u, v \in H$, $\langle Su|v\rangle = \langle u|S^*v\rangle$.
- The linear operator $I.H \to H$ such that for every $v \in H$, $Iv = v$ is called the *identity.*

Remark .2.4. The uniqueness of the adjoint comes from a duality relation between H and H' (see [28], Volume 1, Chapter VI, Section 2).

Fact .2.5. $B(H)$ is a C^*-algebra under the adjunction operation.

Remark .2.6. There are three important topologies on $B(H)$: The norm topology, the *strong* and the *weak*. The strong topology is the topology of pointwise convergence. In weak topology $T_k \to T$ if for all v and $w \in H$, $\langle T_k v \mid w\rangle \to \langle Tv \mid w\rangle$

Definition .2.7. Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ is said to be *positive* if there is $b \in \mathcal{A}$ such that $a = b^*b$. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$. Let a, $b \in \mathcal{A}$. We say that $a \leq b$ if $b - a$ is positive.

Definition .2.8.

- Let $\mathcal{A}$ be a C^*-algebra. A *unit* in $\mathcal{A}$ is an element $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$.
- A C^*-algebra with a unit is called *unital*.
- Let $\mathcal{A}$ be a C^*-algebra. An *approximate unit* in $\mathcal{A}$ is a net $(a_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{A}$ such that $\lambda < \mu$ implies that $a_\lambda \leq a_\mu$ and $\lim \|x - xa_\lambda\| = 0$ for all $x \in \mathcal{A}$.

Fact .2.9 (Theorem 1.4.2 in [31])**.** Each C^*-algebra contains an approximate unit.

Definition .2.10. Let S a bounded linear operator from H to H. Then,

- S is called of *finite rank* if $\dim(SH) < \infty$.
- S is called *compact* if the image under S of the unit ball is relatively compact.
- The set of all compact operators in H is denoted by $\mathcal{K}(H)$.

Fact .2.11 (Theorem VI.13 in [28])**.** A bounded linear operator K is compact if and only if there exists a sequence of finite rank operators F_n such that F_n converges to K in the norm topology.

Fact .2.12. The set $\mathcal{K}(H)$ of all the compact operators on H defines a closed ideal in $B(H)$

Definition .2.13. The algebra $B(H)/\mathcal{K}(H)$ is called the *Calkin algebra* on H. If N is a normal operator on H, the C^*-subalgebra of $B(H)/\mathcal{K}(H)$ generated by equivalence class of N and the equivalence class of I is denoted $\tilde{C}^*(N)$.

Definition .2.14. Given a subset $M \subseteq B(H)$, we define the*commutant* M' of M the set,

$M' = \{S \in B(H) \mid \forall T \in M, ST = TS\}$

Theorem .2.15 (Von Neumann Bicommutant Theorem. Theorem 2.2.2 in [31])**.** *Let M be a subalgebra of $B(H)$ containing the identity. Then the following are equivalent:*

1. *$M = M''$.*
2. *M is weakly closed.*
3. *M is strongly closed.*

Definition .2.16. A C^*-subalgebra of $B(H)$ satisfying any of these equivalent condition is called a *Von Neumann algebra.*

Theorem .2.17 (Kaplansky density theorem. Theorem 2.3.3. in [31])**.** *Let $\mathcal{A}$ be a C^*-subalgebra of $B(H)$ with strong closure M. Then the unit ball $\mathcal{A}^1$ of $\mathcal{A}$ is strongly dense in the unit ball M^1 of M.*

Definition .2.18. Let $\mathcal{A}$, $\mathcal{B}$ be two C^*-algebras. A linear operator $\rho : \mathcal{A} \to \mathcal{B}$ is said to be a **-homomorphism* if

- For all a, $b \in \mathcal{A}$, $\rho(ab) = \rho(a)\rho(b)$
- For all $a \in \mathcal{A}$, $\rho(a^*) = \rho(a)^*$

Definition .2.19. Let $\mathcal{A}$ be a C^*-algebra. An element $p \in \mathcal{A}$ is said to be a *projection* if $p^2 = 2$. Two projections p and $q \in \mathcal{A}$ are said to be *disjoint* if $pq = qp = 0$.

Definition .2.20. A von Neumann algebra M is called *σ-finite* if each set of pairwise orthogonal non-zero projections in M is countable. A projection $p \in M$ is called *σ-finite* if pMp is σ-finite

Definition .2.21. Let M and N von Neumann subalgebras of $B(H)$ and $B(K)$ respectively. A positive linear map $\rho : M \to N$ is is said to be *normal* if for each bounded monotone increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in M_{sa} with limit x, the net $(\rho(x_\lambda))_{\lambda\in\Lambda}$ increases to $\rho(x)$ in N_{sa}.

Definition .2.22.
1. The set $M \cap M'$ is called the *center* of M and is denoted by $Z(M)$.
2. A von Neumann algebra is said to be a *factor* if $Z(M)$ isomorphic to $\mathbb{C}$ as C^*-algebras.

Definition .2.23. Let $x \in M_{sa}$. The *central cover* of x is the infimum of all $z \geq x$ in $Z(M)_{sa}$. This central cover is denoted by $c(x)$.

Definition .2.24. Let $\mathcal{A}$ be a C^*-algebra.

1. A *weight* is a function $\phi : \mathcal{A}_+ \to [0, \infty]$ such that:
   a) $\phi(\alpha x) = \alpha\phi(x)$ for all $x \in \mathcal{A}_+$ and $\alpha \in \mathbb{R}_+$;
   b) $\phi(x + y) = \phi(x) + \phi(y)$ for all x, $y \in \mathcal{A}_+$.
2. A weight ϕ on $\mathcal{A}$ is said to be *densely defined* if the set
   $$\mathcal{A}_+^\phi = \{x \in \mathcal{A}_+ \mid \phi(x) < \infty\}$$
   is dense in $\mathcal{A}_+$.
3. If $\mathcal{A}$ is a von Neumann algebra, $\mathcal{A}$ is said to be *semifinite* if $\mathcal{A}_+^\phi$ is weakly dense in $\mathcal{A}$.
4. If $\mathcal{A}$ is a Borel* algebra, $\mathcal{A}$ is said to be *σ-finite* if $\mathcal{A}_+^\phi$ contains an increasing sequence with limit 1.

5. A weigth ϕ is said to be *lower semicontinuous* if for every $\alpha \in \mathbb{R}_+$, the set $\{x \in \mathcal{A}_+ \mid \phi(x) \leq \infty\}$ is closed.

6. If $\mathcal{A}$ is a Borel*-algebra, a weight ϕ on $\mathcal{A}$ is called σ-*normal* if there exists a sequence ϕ_n of sequentially normal positive functionals such that $\phi = \sum_n \phi_n$.

7. A weight which is σ-finite and σ-normal is called a σ-*weight*.

8. A *trace* on $\mathcal{A}$ is a weight on $\mathcal{A}$ such that $\phi(uxu^*) = \phi(x)$ for all $x \in \mathcal{A}_+$ and $u \in \mathcal{A}$ unitary.

9. If $\mathcal{A}$ is a von Neumann algebra, then:
   a) $\mathcal{A}$ is said to be *finite* if there is a faithful, normal, finite trace on $\mathcal{A}$.
   b) $\mathcal{A}$ is said to be *semifinite* if there is a faithful, normal, semifinite trace on $\mathcal{A}$.
   c) $\mathcal{A}$ is said to be *properly infinite* if there is no nonzero normal, finite trace on $\mathcal{A}$.
   d) $\mathcal{A}$ is said to be *purely infinite* if there is no nonzero normal, semifinite trace on $\mathcal{A}$.

Theorem .2.25 (Proposition 5.4.2 in [31])**.** *Each von Neumann algebra M has a unique decomposition $M = M_1 \oplus M_2 \oplus M_3$ such that M_1 is finite, M_2 is semifinite but properly infinite and M_3 is purely infinite.*

Definition .2.26. Let M be a von Neumann algebra.

1. A projection $p \in M$ is said to be *abelian* if pMp is abelian.

2. M is said to be of *type I* if there exists a abelian projection $p \in M$ such that $c(p) = 1$.

3. M is said to be of *type II* if it is semifinite but doesnot contain any nonzero abelian projection.

4. M is said to be of *type III* if it is purely infinite.

Corollary .2.27. *Each von Neumann algebra M has a unique decomposition $M = M_1 \oplus M_2 \oplus M_3$ such that M_1 is of type I, M_2 is of type II and M_3 is of type III.*

Definition .2.28. Let M be a von Neumann algebra. If M is of type I it is said to be of *type* I_n, where $n = 1, \cdots, \infty$, if there are n disjoint, equivalent, abelian projections $p_1, \cdots, p_n$ in M such that $p_1 + p_2 + \cdots + p_n = 1$.

Theorem .2.29 (Proposition 5.5.7 in [31])**.** *Each von Neumann algebra of type I on a separable Hilbert space H has a unique decomposition $M = \Pi_{n=1}^{\infty} M_n$ where each M_n is of type I_n.*

Theorem .2.30 (Theorem 4.1 in [30])**.** *A factor of type I_∞ is isomorphic to $B(\ell^2)$.*

Definition .2.31. Let M be a type II von Neumann algebra. Then

1. M is called of *type* II_1 if M is finite.
2. M is called of *type* II_∞ if M is properly infinite.

Theorem .2.32 (Theorem 2.3 in [30])**.** *Let M be a factor. There is a unique trace $D : M_+ \to [0,\infty]$ such that*

1. *If p and q are disjoint projections then $D(p+q) = D(p) + D(q)$*
2. *M is a factor of type I_n if and only if the rank of D is $\{0, \dots, n\}$*
3. *M is a factor of type I_∞ if and only if the rank of D is $\{0, \dots, \infty\}$*
4. *M is a factor of type II_1 if and only if the rank of D is $[0, 1]$*
5. *M is a factor of type II_∞ if and only if the rank of D is $[0, \infty]$*
6. *M is a factor of type III if and only if the rank of D is $\{0, 1\}$*

This trace is called the Dimension of M.

Theorem .2.33 (Proposition 5.5.13 in [31])**.** *Each von Neumann algebra on a separable Hilbert space has a unique docomposition:*

$M = M_1 \oplus M_2 \oplus, \cdots, M_\infty \oplus M_{II_1} \oplus M_{II_\infty} \oplus M_{III},$

where the M_n are of type I_n, M_{II_1} is of type II_1, M_{II_∞} is of type II_∞ and M_{III} is of type III.

.3 Representations of C^*-algebras and bounded positive linear functionals

This section deals with the representations of a C^*-algebra and with bounded positive linear functionals. The main theorems here are Theorem .3.30 which gives a canonical way to build representations of a C^*-algebra called the *Gelfand-Naimark-Segal* construction; Theorem .3.36 that generalizes Radon Nikodim Theorem; and Theorem .3.20, which states that a representations of an algebra of compact operators can be seen as a direct sum of representations on finite dimensional Hilbert spaces.

Gelfand-Naimark-Segal construction will be very helpful in defining definable closures and forking between types. Theorem .3.20 we will be used in Section 3.3 to characterize the theory of (H, π). The Gelfand-Naimark-Segal construction and Theorem .3.36 will be used in Section 3.4 to show that positive linear functionals will correspond to types of vectors in H, and in Section 3.8 to prove that the relations of almost domination and orthogonality between positive linear functionals over $\mathcal{A}$ characterize domination and orthogonality between types.

Definition .3.1. Let $\mathcal{A}$ be a C^*-algebra. A *representation* is an algebra homomorphism $\pi : \mathcal{A} \to B(H)$ such that for all $a \in \mathcal{A}$, $\pi(a^*) = (\pi(a))^*$. In this case H is called an $\mathcal{A}$-*module*. A Hilbert subspace $H' \subseteq H$ is called an $\mathcal{A}$-*submodule* or a *reducing* $\mathcal{A}$-*subspace* of H if H' is closed under π. H is called $\mathcal{A}$-*irreducible* or $\mathcal{A}$-*minimal* if H has no proper non trivial $\mathcal{A}$-submodules. The set of representations of an algebra $\mathcal{A}$ on $B(H)$ is denoted $rep(\mathcal{A}, B(H))$.

Definition .3.2. Let (H, π) be a representation of a C^*-algebra $\mathcal{A}$. (H, π) is called *non-degenerate* if for every nonzero vector $v \in H$, there exists $a \in \mathcal{A}$ such that $\pi(a)v \neq 0$.

Fact .3.3 (Remark 2.2.4 in [31])**.** A representation (H, π) of an unital C^*-algebra $\mathcal{A}$ is non-degenerate if and only if $\pi(e) = I$, where e is the identity of $\mathcal{A}$ and I is the identity of $B(H)$.

Assumption .3.4. From now on, every C^*-algebra $\mathcal{A}$ will be assumed to have identity e and every representation will be asumed to be non-degenerate.

Fact .3.5 (Corollary 2.2.5 in [31])**.** Let $\mathcal{A}$ be a C^*algebra and $\pi : \mathcal{A} \to B(H)$ a nondegenerate representation of $\mathcal{A}$. Let $\mathcal{M}$ be the strong closure of $\pi(\mathcal{A})$. Then $\mathcal{M}$ is weakly closed and $\mathcal{M} = \mathcal{A}''$.

Definition .3.6. Two representations (H_1, π_1) and (H_2, π_2) are said to be *unitarily equivalent* if there exists an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$.

Definition .3.7. Two subrepresentations (H_1, π_1), (H_2, π_2) of (H, π) are said to be *disjoint* if no subrepresentation of (H_1, π_1) is unitarily equivalent to any subrepresentation of (H_2, π_2).

Fact .3.8 (Proposition 3 in [12], Chapter 5, Section 2)**.** Two subrepresentations (H_1, π_1), (H_2, π_2) of (H, π) are disjoint if and only if there is a projection P in $\pi(\mathcal{A})' \cap \pi(\mathcal{A})''$ such that if P_1 and P_2 are the projections on H_1 and H_2 respectively, we have that $PP_1 = P_1$ and $(I - P)P_2 = P_2$.

Definition .3.9. (H, π) is called *cyclic* if there exists a vector v_π such that $\pi(\mathcal{A})v_\pi$ is dense in H. Such a vector is called a *cyclic vector* for the representation (H, π).

Remark .3.10. For $v \in H$, it is clear that v is a cyclic vector for $\mathcal{A}$ on H_v.

Notation .3.11. We say that (H, π, v_π) is a *cyclic representation* if v_π is a cyclic vector for (H, π).

Theorem .3.12 (Remark 3.3.1. in [31])**.** *Every representation can be seen as a direct sum of cyclic representations.*

Definition .3.13. Two cyclic representations (H_1, π_1, v_1) and (H_2, π_2, v_2) are said to be *isometrically isomorphic* if there is an isometry U from H_1 to H_2 such that for every $a \in \mathcal{A}$, $U\pi_1(a)U^* = \pi_2(a)$ and $Uv_1 = v_2$.

Theorem .3.14 (Proposition 3.3.7 in [31]). *Two cyclic representations* (H_1, π_1, v_1) *and* (H_2, π_2, v_2) *are isometrically isomorphic if and only if for all* $a \in \mathcal{A}$, $\langle \pi_1(a)v_1 | v_1 \rangle = \langle \pi_2(a)v_2 | v_2 \rangle$.

Definition .3.15. Two representations (H_1, π_1) and (H_2, π_2) are said to be *approximately unitarily equivalent* if there exists a sequence of unitary operators $(U_n)_{n<\omega}$ from H_1 to H_2 such that for every $a \in \mathcal{A}$ $\pi_2(a) = \lim_{n\to\infty} U_n \pi_1(a) U_n^*$ where the limit is taken in the norm topology.

Theorem .3.16 (Theorem II.5.8 in [14]). *Two nondegenerate representations* (H_1, π_1) *and* (H_2, π_2) *of a separable* C^**-algebra on separable Hilbert spaces are approximately unitarily equivalent if and only if, for all* $a \in \mathcal{A}$, *rank*$(\pi_1(a)) =$ *rank*$(\pi_2(a))$

Definition .3.17. A representation (H, π) of $\mathcal{A}$ is called *compact* if $\pi(\mathcal{A}) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ is the algebra of compact operators on H.

Lemma .3.18 (Lemma I.10.7 in [14]). *Let* $\mathcal{A}$ *be an algebra of compact operators on a Hilbert space* H. *Every non-degenerate representation of* $\mathcal{A}$ *is a direct sum of irreducible representations which are unitarily equivalent to subrepresentations of the identity representation.*

Notation .3.19. For a Hilbert space H and a positive integer n, $H^{(n)}$ denotes the direct sum of n copies of H. If $S \in B(H)$, $S^{(n)}$ denotes the operator on $H^{(n)}$ given by $S^n(v_1, \cdots, v_n) = (Sv_1, \cdots, Sv_n)$. If $\mathcal{B} \subseteq B(H)$, $\mathcal{B}^{(n)}$ is the set $\{S^{(n)} \mid S \in \mathcal{B}\}$.

Theorem .3.20 (Theorem I.10.8 in [14]). *Let* (H, π) *be a compact representation of* $\mathcal{A}$. *Then for every* $i \in \mathbb{Z}^+$, *there are a Hilbert spaces* H_i *and positive integers* n_i *and* k_i *such that* $dim(H_i) = n_i$ *and*

$$H \simeq ker(\pi(\mathcal{A})) \oplus \bigoplus_{i\in\mathbb{Z}^+} H_i^{(k_i)}$$

and

$$\pi(\mathcal{A}) \simeq 0 \oplus \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i)^{(k_i)}$$

Remark .3.21. In case that $ker(\mathcal{A}) = 0$, ($\mathcal{A}$ no necessarilly unital) we have that this representation is non-degenerate.

Remark .3.22. Recall that if $R \in \bigoplus_{i\in\mathbb{Z}^+} \mathcal{K}(H_i)^{(k_i)}$, then there is a sequence $(R_i)_{i\in\mathbb{Z}^+}$ such that $R_i \in \mathcal{K}(H_i)^{(k_i)}$ and $R = \sum_{i\in\mathbb{Z}^+} R_i$ in the norm topology. This means, in particular, that $\lim_{i\to\infty} \|R_i\| = 0$.

Remark .3.23. Let (H, π) be a non-degenerate representation of $\mathcal{A}$. By Theorem .3.20, for every $i \in \mathbb{Z}^+$, there are a Hilbert spaces H_i and positive integers n_i and k_i such that $\dim(H_i) = n_i$ and

$$H_d \simeq \bigoplus_{i\in\mathbb{Z}^+} H_i^{(k_i)}$$

Definition .3.24. Let $\mathcal{A}'$ be the dual space of $\mathcal{A}$. An element $\phi \in \mathcal{A}'$ is called *positive* if $\phi(a) \geq 0$ whenever $a \in \mathcal{A}$ is positive, i.e. there is $b \in \mathcal{A}$ such that $a = b^*b$. The set of positive functionals is denoted by $\mathcal{A}'_+$.

Lemma .3.25. *Let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H, and let $v \in H$. Then the function ϕ_v on $\mathcal{A}$ such that for every $S \in \mathcal{A}$, $\phi_v(S) = \langle Sv \,|\, v\rangle$ is a positive linear functional.*

Proof. Linearity is clear. Let S be a positive selfadjoint operator in $\mathcal{A}$, let Q be its square root, that is, an operator such that $S = QQ^*$. Let $v \in H$; then $\langle Sv \,|\, v\rangle = \langle Q^*Qv \,|\, v\rangle = \langle Qv \,|\, Qv\rangle \geq 0$ □

Definition .3.26. Let ϕ be a positive linear functional on $\mathcal{A}$. Let

$$\Lambda^2(\mathcal{A}, \phi) = \{a \in \mathcal{A} \mid \phi(a^*a) < \infty\}/\sim_\phi,$$

where $a_1 \sim_\phi a_2$ if $\phi(a_1^*a_2) = 0$. For $(a)_{\sim_\phi}$, $(b)_{\sim_\phi} \in \Lambda^2(\mathcal{A}, \phi)$, let

$$\langle (a)_{\sim_\phi} \mid (b)_{\sim_\phi}\rangle_\phi = \phi(a^*b).$$

Remark .3.27. The product $\langle \cdot \mid \cdot\rangle_\phi$ is a natural inner product on the space $\Lambda^2(\mathcal{A}, \phi)$(see [12] page 472).

Definition .3.28. We define the space $L^2(\mathcal{A}, \phi)$ to be the completion of $\Lambda^2(\mathcal{A}, \phi)$ under the norm defined by $\langle \cdot \mid \cdot\rangle_\phi$.

Definition .3.29. Let ϕ be a positive linear functional on $\mathcal{A}$. We define the representation $M_\phi : \mathcal{A} \to B(L^2(\mathcal{A}, \phi))$ in the following way: For every $a \in \mathcal{A}$ and $(b)_{\sim_\phi} \in L^2(\mathcal{A}, \phi)$, let $M_\phi(a)((b)_{\sim_\phi}) = (ab)_{\sim_\phi}$.

Theorem .3.30 (Theorem 3.3.3. and Remark 3.4.1. in [31])**.** *Let ϕ be a positive functional on $\mathcal{A}$. Then there exists a cyclic representation $(H_\phi, \pi_\phi, v_\phi)$ such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a)v_\phi | v_\phi\rangle$. This representation is called the Gelfand-Naimark-Segal construction.*

Proof. Take $(L^2(\mathcal{A}, \phi_v), M_{\phi_v}, (e)_{\sim_{\phi_v}})$. Note that

$$\langle M_{\phi_v}(a)(e)_{\sim_{\phi_v}} \mid (e)_{\sim_{\phi_v}}\rangle = \langle (a)_{\sim_{\phi_v}} \mid (e)_{\sim_{\phi_v}}\rangle = \phi_v(a \cdot e) = \phi_v(a).$$

□

Theorem .3.31. *Let $v \in H$. Then $(H_v, \pi_v, v) \simeq (L^2(\mathcal{A}, \phi_v), M_{\phi_v}, (e)_{\sim_{\phi_v}})$.*

Proof. By Gelfand-Naimark-Segal Theorem .3.30 and Theorem .3.14. □

Definition .3.32. We define the following (see [31]):

1. A positive linear functional ϕ on $\mathcal{A}$ is called a *quasistate* if $\|\phi\| \leq 1$.

2. The set of the of quasistates on $\mathcal{A}$ is denoted by $Q_{\mathcal{A}}$.
3. In the case where $\|\phi\| = 1$, the positive linear functional ϕ is called a *state*.
4. The set of states is denoted by $S_{\mathcal{A}}$.
5. A state is called *pure* if it is not a convex combination of other states.
6. The set of pure states is denoted by $PS_{\mathcal{A}}$.

Definition .3.33. Let ϕ and ψ be positive linear functionals on $\mathcal{A}$.

1. They are called *orthogonal* ($\phi \perp \psi$) if $\|\phi - \psi\| = \|\phi\| + \|\psi\|$.
2. Also, ϕ is called *dominated* by ψ ($\phi \leq \psi$) if there exist $\gamma > 0$ such that the functional $\gamma\psi - \phi$ is positive.

Fact .3.34 (Lemma 3.2.3 in [31])**.** Let ϕ and ψ be two positive linear functionals on $\mathcal{A}$. Then, $\phi \perp \psi$ if and only if for all $\epsilon > 0$ there exists a positive element $a \in \mathcal{A}$ with norm less than or equal to 1, such that $\phi(e - a) < \epsilon$ and $\psi(a) < \epsilon$.

Lemma .3.35. *Let ϕ_1, ϕ_2, ψ_1 and ψ_2 be positive linear functionals on $\mathcal{A}$ such that $\phi_1 \leq \phi_2$ and $\psi_1 \leq \psi_2$. If $\phi_2 \perp \psi_2$, then $\phi_1 \perp \psi_1$.*

Proof. Let $\gamma_1 > 0$ and $\gamma_2 > 0$ be such that $\gamma_1\phi_2 - \phi_1$ and $\gamma_2\psi_2 - \psi_1$ are positive. By Fact .3.34, for $\epsilon > 0$ there exists a positive $a \in \mathcal{A}$ with norm less than or equal to 1 such that $\phi_2(e - a) < \frac{\epsilon}{\gamma_1+\gamma_2}$ and $\psi_2(a) < \frac{\epsilon}{\gamma_1+\gamma_2}$. Then $\phi_1(e - a) \leq \gamma_1\phi_2(e - a) < \frac{\gamma_1\epsilon}{\gamma_1+\gamma_2} < \epsilon$ and $\psi_1(a) \leq \gamma_2\psi_2(a) < \frac{\gamma_2\epsilon}{\gamma_1+\gamma_2} < \epsilon$. □

Theorem .3.36 (Generalized Radon-Nikodim Theorem in [16])**.** *Let $\pi : \mathcal{A} \to B(H)$ be a representation and let v, $w \in H$. Then $\phi_v \leq \phi_w$ if and only if there exists a bounded positive operator $P : H_w \to H_v$ that commutes with $\pi(\mathcal{A})$ and $P(w) = v$.*

Definition .3.37. Let (H_i, π_i) for $i \in I$ be a family of representations of $\mathcal{A}$. We define a representation $\oplus\pi_i$ on $\oplus H_i$ in the following way: Let $v = \sum_i v_i$ and $a \in \mathcal{A}$, $\oplus\pi_i(a)v = \sum_i \pi_i(a)v_i$.

Definition .3.38. We define the following:

- A subset $F \subseteq S_{\mathcal{A}}$ is called *separating* if for every $a \in \mathcal{A}$, $\phi(a) = 0$ for every $\phi \in F$ implies that $a = 0$.
- Let $\phi \in S_{\mathcal{A}}$. ϕ is said to be *faithful* if for every $a \in \mathcal{A}_+$, $\phi(a) = 0$ implies that $a = 0$. A *faithful representation* is a representation (H, π) such that if $\pi(a) = 0$ then $a = 0$ for $a \in \mathcal{A}_+$.

Notation .3.39. For each $\phi \in S_{\mathcal{A}}$, let (H_ϕ, π_ϕ) be the Gelfand-Naimark-Segal construction of ϕ. For $F \subseteq S_{\mathcal{A}}$ let $(H_F, \pi_F) = (\oplus_{\phi \in F} H_\phi, \oplus_{\phi \in F} \pi_\phi)$.

Theorem .3.40 (Proposition 3.7.4 in [31])**.** *If $F \subseteq S_{\mathcal{A}}$ is separating, then (H_F, π_F) is a faithful representation.*

Definition .3.41. Let $H_{S_{\mathcal{A}}}$ be the space,

$$H_{S_{\mathcal{A}}} = \oplus_{\phi \in S_{\mathcal{A}}} L^2(\mathcal{A}, \phi)$$

and let $\pi_{S_{\mathcal{A}}}$ be,

$$\pi_{S_{\mathcal{A}}} = \oplus_{\phi \in S_{\mathcal{A}}} M_\phi,$$

Definition .3.42. The representation $(H_{S_{\mathcal{A}}}, \pi_{S_{\mathcal{A}}})$ is called the *universal representation.*

.4 Spectral theory of a closed unbounded self-adjoint operator

This is a small review of spectral theory of a closed unbounded self-adjoint operator. The main sources for this section are [29, 28].
In the next definition we broaden the definition of linear operator. In this case, we allow the domain not to be all H but a dense subset of H:

Definition .4.1. Let H be a complex Hilbert space. A *linear operator on H* is a function $S : D(S) \to H$ such that $D(S)$ is a dense vector subspace of H and for all v, $w \in S$ and α, $\beta \in \mathbb{C}$, $S(\alpha v + \beta w) = \alpha S v + \beta S w$.

Definition .4.2. Let S be a linear operator on H. The operator S is called *bounded* if the set $\{\|Su\| : v \in D(S), \|v\| = 1\}$ is bounded in $\mathbb{C}$. If S is not bounded, it is called *unbounded.*

Definition .4.3. If S is bounded we define the *norm* of S by:

$$\|S\| = \sup_{u \in D(S), \|u\|=1} \|Su\|$$

For H a Hilbert space, we denote by $B(H)$ the algebra of all bounded linear operators on H such that $D(S) = H$.

Definition .4.4. Let R and S be linear operators on H and let $\alpha \in \mathbb{C}$. Then the linear operators $R + S$, αS and S^{-1} are defined as follows:

1. If $D(R) \cap D(S)$ is dense in H, $D(R + S) := D(R) \cap D(S)$ and $(R + S)v := Rv + Sv$ for $v \in D(R + S)$.

2. $D(RS) := \{v \in H \mid v \in D(S) \text{ and } Sv \in D(R)\}$, $(RS)v := R(Sv)$ if $D(RS)$ is dense and $v \in D(RS)$.

3. If $\alpha = 0$, then $\alpha T \equiv 0$ in H. If $\alpha \neq 0$, $D(\alpha S) := D(S)$ and $(\alpha S)v := \alpha Sv$ if $v \in D(S)$

4. If S is one-to-one and $SD(S)$ is dense in H, $D(S^{-1}) := SD(S)$ and $S^{-1}v := w$ if $w \in D(S)$ and $Sw = v$

Definition .4.5. Let $S : D(S) \to H$ be a linear operator on H. The operator S is called *closed* if the set $\{(v, Sv) \mid v \in D(S)\}$ is closed in $H \times H$. The operator S is called *closable* if the closure of the set $\{(v, Sv) \mid v \in D(S)\}$ is the graph of some operator which is called the *closure* of S and is denoted by $\bar{S}$.

Definition .4.6. Given linear operators $S : D(S) \to H$ and $S' : D(S') \to H$ on H, S' is said to be an *adjoint operator* of S if for every $v \in D(S)$ $w \in D(S')$, $\langle Sv|w\rangle = \langle v|S'w\rangle$.

Definition .4.7. Given a linear operator $S : D(S) \to H$ and $S' : D(S') \to H$ on H, then S' is said to be the *adjoint operator* of S, denoted S^*, if S' is maximal adjoint to S i.e. if S'' is and adjoint operator of S and $S' \subseteq S''$ then $S' = S''$.

Remark .4.8. Existence of adjoint operator comes from Riesz Representation Theorem and the uniqueness is a consequence of the density of $D(S)$.

Definition .4.9. A linear operator Q on H is called *symmetric* if $Q \subseteq Q^*$. If $Q = Q^*$, Q is called *self-adjoint*.

Example .4.10 (Example 3 in Section VIII.1 in [28]). Let $H = L^2(\mathbb{R})$, $D(S) = C_0^\infty(\mathbb{R})$. Let $S := i\frac{d}{dx}$. Then S is symmetric, and closable.

Definition .4.11. Let S be an operator (either bounded or unbounded), and λ a complex number. Then,

1. λ is called a *eigenvalue* of S if the operator $S - \lambda I$ is not one to one. The *point spectrum* of S, denoted by $\sigma_p(S)$, is the set of all the eigenvalues of S.

2. λ is called a *continuous spectral value* if the operator $S - \lambda I$ is one to one, the operator $(S - \lambda I)^{-1}$ is densely defined but is unbounded. The *continuous spectrum* of S ($\sigma_c(S)$) is the set of all the continuous spectral values of S.

3. λ is called a *residual spectral value* if $(S - \lambda I)H$ is not dense in H. The *residual spectrum* of S ($\sigma_r(S)$) is the set of all the residual spectral values of S.

4. The *spectrum* of S ($\sigma(S)$) is the union of $\sigma_p(S)$, $\sigma_c(S)$ and $\sigma_r(S)$.

5. The *resolvent set* of S ($\rho(S)$) is the set $\mathbb{C} \setminus \sigma(S)$.

6. If $\lambda \in \rho(S)$, the *resolvent operator of* S *at* λ is the operator $(S-\lambda I)^{-1}$, and is denoted by $R_\lambda(S)$.

Fact .4.12 (Lemma XII.2.2 in [29])**.** The spectrum of a self-adjoint operator Q is real and for $\lambda \in \rho(Q)$, the resolvent $R_\lambda(Q)$ is a normal operator with $R_\lambda(Q)^* = R_{\bar\lambda}(Q)$ and $\|R_\lambda(Q)\| \leq |Im(\lambda)|$.

Fact .4.13 (Theorem VIII.1 in [28])**.** Let Q be an operator on H. Then,

- Q^* is closed.
- Q is closable if and only if $D(Q^*)$ is dense in H in which case $\bar{Q} = Q^{**}$.
- If Q is closable then $(\bar{Q})^* = Q^*$.

Fact .4.14 (Theorem VIII.3 in [28])**.** Let Q be a symmetric operator on H. Then the following statements are equivalent:

1. Q is self-adjoint.
2. Q is closed and $Ker(Q^* \pm iI) = \{0\}$.
3. $Ran(Q \pm iI) = H$.

Definition .4.15. A symmetric operator S is called *essentially self-adjoint* if its closure $\bar{S}$ is self-adjoint.

Fact .4.16 (Corollary of Theorem VIII.3 in [28])**.** Let Q be a symmetric operator on H. Then the following statements are equivalent:

1. Q is essentially self-adjoint.
2. $Ker(Q^* \pm iI) = \{0\}$.
3. $Ran(Q \pm iI)$ is dense.

Fact .4.17 (Theorem 9.1-2 in [26])**.** Let $Q : H \to H$ be a closed self-adjoint operator on H. Then a number $\lambda \in \mathbb{R}$ belongs to $\sigma(Q)$ if and only if there exists $c > 0$ such that for every $v \in D(Q)$, $\|(Q-\lambda I)v\| \geq c\|v\|$.

Remark .4.18. The Previous theorem was originally stated for bounded operators, but its generalization to closed unbounded self-adjoint operators is straightforward and left to the reader. Recall that $\sigma(Q) \subseteq \mathbb{R}$ by Fact .4.12.

Theorem .4.19 (Spectral Theorem Multiplication Form, Theorem VIII.4 in [28])**.** *Let* Q *be self-adjoint on a Hilbert space* H *with domain* $D(Q)$*. Then there are a measure space* (X, μ)*, with* μ *finite, an unitary operator* $U : H \to L^2(X, \mu)$*, and a real function* f *on* X *which is finite a.e. so that,*

1. *$v \in D(Q)$ if and only if $f(\cdot)(Uv)(\cdot) \in L^2(X, \mu)$.*
2. *If $g \in U(D(Q))$, then $(UQU^{-1}g)(x) = f(x)g(x)$ for $x \in X$.*

Definition .4.20. A self-adjoint operator Q different from the zero operator is called *positive* and we write $Q \geq 0$, if $\langle Qv|v\rangle \geq 0$ for all $v \in \mathcal{H}$.

Theorem .4.21 (Spectral Theorem-Functional Calculus Form, Theorem VIII.5 in [28])**.** *Let Q be a closed unbounded self-adjoint operator on H. Then there is a unique map π from the bounded Borel functions on $\mathbb{R}$ into $B(H)$ such that,*

1. *π is an algebraic *-homomorphism.*
2. *π is norm continuous, that is, $\|\pi(h)\|_{B(H)} \leq \|h\|_\infty$.*
3. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions with $h_n(x) \to x$ for each x and $|h_n(x)| \leq |x|$ for all x and n. Then for any $v \in D(Q)$, $\lim_{n\to\infty} \pi(h_n)v = Qv$.*
4. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions. If $h_n \to h$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\pi(h_n) \to \pi(h)$ strongly.*
5. *If $v \in H$ is such that $Qv = \lambda v$, then $\pi(h)v = h(\lambda)v$.*
6. *If $h \geq 0$, then $\pi(h) \geq 0$*

Definition .4.22. Let Ω be a Borel measurable subset of $\mathbb{R}$. By E_Ω we denote the bounded operator $\pi(\chi_\Omega)$ according to Theorem .4.21.

Fact .4.23 (Remark after Theorem VIII.5 in [28])**.** Previously defined projections satisfy the following properties:

1. For every Borel measurable $\Omega \subset \mathbb{R}$, $E_\Omega^2 = E_\Omega$ and $E_\Omega^* = E_\Omega$.
2. $E_\emptyset = 0$ and $E_{(-\infty,\infty)} = I$
3. If $\Omega = \cup_{n=1}^\infty \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $\sum_{n=1}^\infty E_{\Omega_n}$ converges to E_Ω in the strong topology.
4. $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1\cap\Omega_2}$ (and therefore E_{Ω_1} commutes with E_{Ω_2}) for all Borel measurable Ω_1, $\Omega_2 \subseteq \mathbb{R}$.

Definition .4.24. The family $\{E_\Omega \mid \Omega \subseteq \mathbb{R}$ is Borel measurable $\}$ described in Fact .4.23 is called the *spectral projection valued measure* (s.p.v.m.) generated by Q.

Fact .4.25 (Remark before Theorem VIII.6 in [28])**.** Let $v \in \mathcal{H}$. Then the set function such that for every Borel set $\Omega \subset \mathbb{R}$ assigns the value $\langle E_\Omega v|v\rangle$ is a Borel measure. In the case when $\Omega = (-\infty, \lambda)$, this measure is denoted $\langle E_\lambda v|v\rangle$.

Fact .4.26 (Spectral Theorem-Integral Decomposition form, Theorem VIII.6 in [28])**.** Let Q be a closed unbounded self-adjoint operator on H and let h be a (possibly unbounded) Borel measurable function on $\mathbb{R}$. Then the (possibly unbounded) operator $h(Q)$ defined as the only operator such that

$$\langle h(Q)v \, |v\rangle := \int_{-\infty}^{\infty} h(l) d\langle E_\lambda v \mid v\rangle,$$

whenever $v \in D(h(Q))$, with

$$D(h(Q)) := \{v \in h \mid \int_{-\infty}^{\infty} |h(l)|^2 d\langle E_\lambda v \mid v\rangle < \infty\},$$

is such that $h(Q)$ satisfies properties 1-4 of Theorem .4.21 and if h is a bounded Borel measurable function on $\mathbb{R}$, then $h(Q)$ is exactly the operator $\pi(h)$ described in Theorem .4.21.

Definition .4.27. The *essential spectrum* of a closed unbounded self adoint operator Q, denoted by $\sigma_e(Q)$, is the set of complex values λ such that for every bounded operator S on H and every compact operator K on H, we have that $(Q - \lambda I)S \neq I + K$.

Remark .4.28. Let Q be a closed unbounded self-adjoint operator on H. Then $\sigma_e(Q) \subseteq \sigma(Q)$.

Next theorem is known as *Weyl's Criterion.* It gives a useful tool to identify the essential spectrum:

Theorem .4.29 (Weyl's Criterion)**.** *Let Q be a closed unbouned self-adjoint operator. Then, for every $\lambda \in \mathbb{R}$, the following conditions are equivalent:*

i) $\lambda \in \sigma_e(Q)$.

ii) For every $\epsilon > 0$, $\dim(E_{(\lambda-\epsilon,\lambda+\epsilon)}H) = \infty$.

Proof. **(i)⇒(ii)** Asume that there exists $\epsilon > 0$ such that $E_{(\lambda-\epsilon,\lambda+\epsilon)}H$ finite dimensional. Let

$$h(x) = \frac{1 - \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(x)}{x - \lambda}.$$

Then h is a bounded borel measurable function on $\mathbb{R}$. By Fact .4.21 (functional calculus), we have that,

$$h(Q)(Q - \lambda I) = (Q - \lambda I)h(Q) = I - \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q) = I - E_{(l-\epsilon,\lambda+e)}H$$

Since $E_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)$ is finite dimensional, it is compact and $\lambda \notin \sigma_e(Q)$

(ii)⇒(i) Suppose that $\lambda \notin \sigma_e(Q)$. Then there are a bounded operator S and a compact operator K such that,

$$S(Q-\lambda I) = (Q-\lambda I)S = I + K \tag{-1}$$

Suppose that for some $v \in H$, $(Q-\lambda I)v = 0$. Then $(I-K)v = 0$ and, therefore, $Kv = -v$. Since K is compact, this implies that $Ker(Q-\lambda I)$ is finite dimensional By Hypotesis, for all $\epsilon > 0$, $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)$ is infinite dimensional and contains $ker(Q-\lambda I)$ which is finite dimensional. So, for every $\epsilon > 0$ there exists $v_\epsilon \in \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)$ such that $\|v_\epsilon\| = 1$ and $d(v_\epsilon, ker(Q-\lambda I)) = 1$ By Theorem .4.26

$$\|(Q-\lambda I)v_\epsilon\|^2 = \langle (Q-\lambda I)^*(Q-\lambda I)\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(Q)(v_\epsilon)|v_\epsilon\rangle =$$
$$= \int_{\lambda-\epsilon}^{\lambda+\epsilon} |x-\lambda|^2 d\langle E_x v_\epsilon \mid v_\epsilon\rangle \leq \int_{\lambda-\epsilon}^{\lambda+\epsilon} |x-\lambda|^2 dx \leq \epsilon^2 \int_{\lambda-\epsilon}^{\lambda+\epsilon} dx \leq 2\epsilon^3$$

and hence $Qv_\epsilon - \lambda v_\epsilon \to 0$ when $\epsilon \to 0$. From (-1) we get:

$$v_\epsilon + kv_\epsilon = S(Qv_\epsilon - \lambda v_\epsilon) \to 0 \text{ when } \epsilon \to 0.$$

By compactness of k, there exists a sequence $(v_n) \subseteq \{v_\epsilon \mid \epsilon > 0\}$ such that $kv_n \to v$ when $n \to \infty$ for some $v \in H$. It follows that $v_n \to -v$ and, since $\|v_n\| = 1$, we get $\|v\| = 1$. Since $Q(v_n) - \lambda v_n \to 0$ when $n \to \infty$, we get $Qv = \lambda v$, and hence:

$$\|v_n - v\| \geq d(v_n, ker(Q-\lambda I)) = 1,$$

which is a contradiction.

□

Definition .4.30. Let Q be a closed unbounded self-adjoint operator on H. The *discrete spectrum* of Q is the set:

$$\sigma_d(Q) := \sigma(Q) \setminus \sigma_e(Q)$$

Definition .4.31. Let Q_1 and Q_2 be closed unbounded self-adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. Then (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) are said to be *spectrally equivalent* ($Q_1 \sim_\sigma Q_2$) if both of the following conditions hold:

1. $\sigma(Q_1) = \sigma(Q_2)$.
2. $\sigma_e(Q_1) = \sigma_e(Q_2)$.
3. $\dim\{x \in H_1 \mid Q_1 x = \lambda x\} = \dim\{x \in H_2 \mid Q_2 x = \lambda x\}$ for $\lambda \in \sigma(Q_1) \setminus \sigma_e(Q_1)$.

Fact .4.32 (Classical Weyl theorem, Example 3 of Section XIII.4 in [28])**.** If Q is a (possibly unbounded) self-adjoint operator and K is a compact operator on H. Then $\sigma_e(Q) = \sigma_e(Q + K)$.

Fact .4.33 (Weyl-von Neumann-Berg, Corollary 2 in [9])**.** Let Q be a not necessarilly bounded self-adjoint operator on a separable Hilbert space H. Then for every $\epsilon > 0$ there exists a diagonal operator D and a compact operator K on H such that $\|K\| < \epsilon$ and $Q = D + K$.

Definition .4.34. Two unbounded closed self-adjoint operators Q_1 and Q_2 on a separable Hilbert spaces H_1 and H_2 are said to be *approximately unitarily equivalent* if there exists a sequence of unitary operators $(U_n)_{n<\omega}$ from H_1 to H_2 such that for every $n \in \mathbb{Z}_+$, $Q_2 - U_n Q_1 U_n^*$ is bounded and for all $\epsilon > 0$, there is n_ϵ such that for every $n \geq n_\epsilon$, $\|Q_2 - U_n Q_1 U_n^*\| < \epsilon$.

Next theorem is an important consequence of Weyl-von Neumann-Berg Theorem. In Section 4.5, we will give a model theoretic proof of it:

Fact .4.35 (See II.4.4 in [14])**.** Suppose Q_1 and Q_2 are unbounded closed self-adjoint operators on a separable Hilbert space H. Then Q_1 and Q_2 are approximately unitarily equivalent if and only if $Q_1 \sim_\sigma Q_2$.

Definition .4.36. Let $v \in H$, the *spectral measure defined by* v (denoted by μ_v) is the finite Borel measure that to any Borel set $\Omega \subseteq \mathbb{R}$ assigns the (complex) number,

$$\mu_v(\Omega) := \langle \chi_\Omega(Q)v \mid v \rangle$$

Fact .4.37 (Lemma XII.3.1 in [29])**.** For $v \in H$, the space $H_v \simeq L^2(\mathbb{R}, \mu_v)$.

Fact .4.38 (Lemma XII.3.2 in [29])**.** There is a set $G \subseteq H$ such that $H = \oplus_{v \in G} H_v$.

.5 Closed *-representations of *-algebras

In this section we give the main concepts and results we need about *-algebras and *-representations. The main source for this is the book of Schmudgen [32]. As te reader can guess, most of results are generalizations of known results for C^* and von Neumann algebras.

Definition .5.1. Let V be a complex vector space. A *locally convex topology* on V is a topology generated by a family $\mathbb{U}$ of neighbourhoods of the 0 vector in V such that:

- For U_1, $U_2 \in \mathbb{U}$, there is $U \in \mathbb{U}$ such that $U \subseteq U_1 \cap U_2$.
- If $U \in \mathbb{U}$, then $\lambda U \in \mathbb{U}$ for all $\lambda \in \mathbb{C}$ $\lambda \neq 0$.
- Every $U \in \mathbb{U}$, U is convex.
- For every $v \in V$ and every $U \in \mathbb{U}$, there exists $\lambda \in \mathbb{C}$ such that $v \in \lambda U$.

Fact .5.2. Let $\mathfrak{A}$ be a family of seminorms on a vector space V. Then the family:

$\tau_{\mathfrak{A}} := \{\{v \in V \mid f(x) < \epsilon\} \mid \epsilon > 0, f \in \mathfrak{A}\}$

defines a locally convex topology on V. This topology is called the *locally convex topology defined by* $\mathfrak{A}$.

Definition .5.3. Let $\mathcal{A}$ be a complex algebra. $\mathcal{A}$ is called a **-algebra* if there exists a map $^* : \mathcal{A} \to \mathcal{A}$, called *involution* such that for all a, $b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

1. $(a+b)^* = a^* + b^*$
2. $(ab)^* = b^* a^*$
3. $(\alpha a)^* = \bar{\alpha} a^*$
4. $(a^*)^* = a$

and $\mathcal{A}$ has a locally convex topology τ such that multiplication and involution in $\mathcal{A}$ are continuous under τ.

Definition .5.4. Let $a \in \mathcal{A}$. Then,

1. a is called *selfadjoint* if $a = a^*$.
2. a is called *normal* if $aa^* = a^*a$.
3. a is called *unitary* if $a^{-1} = a^*$.

Definition .5.5.
- Let $\mathcal{A}$ be a *-algebra. A *unit* in $\mathcal{A}$ is an element $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$.
- A *-algebra with a unit is called *unital.*

Notation .5.6. From now on, $\mathcal{A}$ will denote a unital *-algebra unless stated otherwise.

Definition .5.7. An element $\phi \in \mathcal{A}$ in the dual of $\mathcal{A}$ is called *positive* if $\phi(a^*a) \geq 0$ whenever $a \in \mathcal{A}$.

Definition .5.8. A positive linear functional ϕ on $\mathcal{A}$ is called a *state* if $\phi(e) = 1$. The set of the states on $\mathcal{A}$ is denoted by $S_{\mathcal{A}}$.

Definition .5.9. A state is called *pure* if it is not a proper convex combination of other states. The set of the pure states on $\mathcal{A}$ is denoted by $PS_{\mathcal{A}}$.

Definition .5.10. Let $S : D(S) \to H$ be a linear operator on H. The operator S is called *closed* if the set $\{(v, Sv) : v \in D(S)\}$ is closed in $H \times H$. The operator S is called *closable* if the closure of the set $\{(v, Sv) : v \in D(S)\}$ is the graph of some operator $\bar{S}$, which is called the *closure* of S.

Definition .5.11. Let D be a dense subspace of a Hilbert space H. Then $\mathcal{L}(D)$ denotes the algebra of all closable operators from D to D.

Definition .5.12. An O^*-algebra $\mathcal{M}$ on a dense subspace $D \subseteq H$, is a complex *-algebra such that:

- $\forall S \in \mathcal{M}$, $S : D \to H$ is a closable linear operator
- $\forall S \in \mathcal{M}$, $S(D) \subseteq D$
- I_D is in $\mathcal{M}$
- $\forall S \in \mathcal{M}$, $D \subseteq D(S^*)$
- $\forall S \in \mathcal{M}$, $S^+ := S^* \upharpoonright D \in \mathcal{M}$ i.e. for every $S \in \mathcal{M}$, S^* the adjoint of S is well defind on D.

If $\mathcal{M}$ is an O^*-algebra on D, the space D is called the *domain* of $\mathcal{M}$ and is denoted by $D(\mathcal{M}) := D$. The Hilbert space H is denoted by $H(\mathcal{M})$.

Notation .5.13. As it can be noticed, $\mathcal{M}$ and $\mathcal{A}$ denote *-algebras. However, $\mathcal{M}$ is used for concrete algebras i.e. made up by operators on some Hilbert space, while $\mathcal{A}$ is commited for abstract ones.

Definition .5.14. We define the algebra $\mathcal{L}^\dagger(D)$ as the set of all linear operators S on H such that:

- $D \subseteq D(S)$
- $S(D) \subseteq D$
- $D \subseteq D(S^*)$
- $S^*(D) \subseteq D$

Remark .5.15. As we stated it in Definition .5.12, if $S \in \mathcal{L}^\dagger(D)$ we define $S^\dagger := S^* \upharpoonright D$.

Fact .5.16 (Proposition 2.1.8 in [32])**.** $\mathcal{L}^\dagger(D)$ is the largest O^*-algebra on the domain D.

Definition .5.17. Let $\mathcal{M}$ be an O^*-algebra on D. The *adjoint* O^*-algebra of $\mathcal{M}$ is the O^*-algebra $\mathcal{M}^*$ such that:

- $D(\mathcal{M}^*) = D^*(\mathcal{M}) = \cap_{S \in \mathcal{M}} D(S^*)$
- For all $R \in \mathcal{M}^*$, there is an $S \in \mathcal{M}$ such that $R = S^* \upharpoonright D(\mathcal{M}^*)$.

If $\mathcal{M}$ is such that $D(\mathcal{M}) = D(\mathcal{M}^*)$ and $\mathcal{M} = \mathcal{M}^*$, we say that $\mathcal{M}$ is *selfadjoint*.

Definition .5.18. Let $\mathcal{M}$ be an O^*-algebra on D. The *graph topology* is the locally convex topology on D defined by the family of seminorms $\{\|\cdot\|_S := \|S\cdot\|\}$.

Remark .5.19. Let $\mathcal{M}$ be an O^*-algebra on D. For $S \in \mathcal{M}$, by $\bar{S}$ we denote de closure of the operator S. By $\bar{\mathcal{M}}$ we denote the algebra:

$$\bar{\mathcal{M}} := \{\bar{S} \mid S \in \mathcal{M}\}$$

Definition .5.20. Let $\mathcal{M}$ be an O^*-algebra on D. The *graph topology* is the locally convex topology on D defined by the family of seminorms $\{\|\cdot\|S := \|S\cdot\|\}$. When D is considered with the graph topology, it is denoted by $D_{\mathcal{M}}$.

Definition .5.21. Let D be a dense subspace of H. The *bounded topology* on $\mathcal{L}^{\dagger}(D)$ is the locally convex topology defined by the family of seminorms:

$$p_{E,F}(S) \sup_{v\in E} \sup_{w\in F} |\langle Sv \mid w\rangle|,$$

where, E and F are bounded subsets of D.

Definition .5.22. Let $\mathcal{M}$ be a O^*-algebra on D. We define $\bar{D}(\mathcal{M}) := \cap_{S\in\mathcal{M}} D(\bar{S})$ and $\bar{\mathcal{M}} := \{\bar{S} \restriction \bar{D}(\mathcal{M}) \mid S \in \mathcal{M}\}$

Definition .5.23. Let $\hat{D}(\mathcal{M})$ denote the closure of $D(\mathcal{M})$ in the locally convex space $\bar{D}(\mathcal{M})_{\bar{\mathcal{M}}}$. The *closure* of $\mathcal{M}$ is the algebra:

$$\hat{\mathcal{M}} := \{\hat{S} := \bar{S} \restriction \hat{D}(\mathcal{M}) \mid S \in \mathcal{M}\}$$

Definition .5.24. Let $\mathcal{A}$ be a *-algebra. A **-representation* of $\mathcal{A}$ on a dense linear suspace D of H is a *-algebra morphism $\pi : \mathcal{A} \to \mathcal{L}(D)$ such that:

- For every α_1, $\alpha_2 \in \mathbb{C}$, a_1, $a_2 \in \mathcal{A}$,

  $$\pi(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1\pi(a_1) + \alpha_2\pi(a_2)$$

- If e is the identity element in $\mathcal{A}$, $\pi(e) \restriction D \equiv I_D$.
- For every $a \in \mathcal{A}$, $\pi(a)D \subseteq D$
- For every $a \in \mathcal{A}$, $\pi(a)$ is a closable linear operator on D
- For every $a \in \mathcal{A}$, $\pi(a) \in \mathcal{L}^{\dagger}(D)$
- For every $a \in \mathcal{A}$, $\pi(a^+) = \pi(a)^+$

Definition .5.25. Let $\pi : \mathcal{A} \to \mathcal{L}(D)$ be a *-representation of a *-algebra $\mathcal{A}$. We denote $\hat{\pi}$ the *-representation such that for all $a \in \mathcal{A}$, $\hat{\pi}(a) = \bar{\pi}(a) \restriction \hat{D}(\pi(\mathcal{A}))$. This *-representation is called the *closure* of π. If $\hat{\pi}(\mathcal{A}) = \pi(\mathcal{A})$, the representation π is said to be *closed*.

Definition .5.26. Let $\pi : \mathcal{A} \to \mathcal{L}(D)$ be a *-representation of a *-algebra $\mathcal{A}$. We denote π^* the *-representation such that for all $a \in \mathcal{A}$, $\pi^*(a) = (\pi^*(a)) \upharpoonright D^*(\pi(\mathcal{A}))$. This *-representation is called the *adjoint* of π. If $\pi^*(\mathcal{A}) = \pi(\mathcal{A})$, the representation π is said to be *selfadjoint*.

Remark .5.27. From now on, we will assume that all of the representations we talk about are self adjoint.

Definition .5.28. Two *-representations π_1 and π_2 are said to be *unitarily equivalent* if there exists an isometry U from $H(\pi_1)$ to $H(\pi_2)$ such that, $UD(\pi_1) \subseteq D(\pi_2)$ and for every $a \in \mathcal{A}$, we have that $U\pi_1(a)U^* = \pi_2(a)$ and for every $v \in D(\pi_1)$, $U\pi_1(a)v = \pi_2(a)Uv$.

Definition .5.29. Let $(\pi_i)_{i\in I}$ be a family of *-representations of $\mathcal{A}$, $\pi_i : \mathcal{A} \to \mathcal{L}(D_i)$, $D_i \subseteq H_i$. We define the *direct sum* $(\bigoplus_{i\in I} \pi_i)$ of $(\pi_i)_{i\in I}$ in the following way:

$$D(\oplus_{i\in I}\pi_i) := \bigoplus_{i\in I} D(\pi_i)$$

$$H(\oplus_{i\in I}\pi_i) := \bigoplus_{i\in I} H(\pi_i)$$

$$(\bigoplus_{i\in I} \pi_i)(a)(\sum_{i\in I} v_i) := \sum_{i\in I} \pi_i(a)(v_i)$$

Definition .5.30. Let $\pi : \mathcal{A} \to \mathcal{L}(D)$. A vector $v \in D(\pi)$ is said to be *algebraically cyclic* if $\pi(\mathcal{A})v = D(\pi(\mathcal{A}))$. If $\pi(\mathcal{A})v$ is dense in $D(\pi)_{\pi(\mathcal{A})}$, v is said to be *cyclic*.

Theorem .5.31 (Theorem 8.6.2. in [32])**.** *Let ϕ be a positive linear functional on $\mathcal{A}$. Then there exists an algebraic cyclic *representation π_ϕ with a cyclic vector $v_\phi \in D(\pi_\phi)$ such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a)v_\phi | v_\phi\rangle$. If π' is another algebraic cyclic *-representation such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi'(a)v_\pi | v_\pi\rangle$ then π' is unitarily equivalent to π_ϕ.*

Theorem .5.32 (Theorem 8.6.4. in [32])**.** *Let ϕ be a positive linear functional on $\mathcal{A}$. Then there exists a cyclic *representation π_ϕ with a cyclic vector $v_\phi \in D(\pi_\phi)$ such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi_\phi(a)v_\phi | v_\phi\rangle$. If π' is another cyclic *-representation such that for all $a \in \mathcal{A}$, $\phi(a) = \langle \pi'(a)v_\pi | v_\pi\rangle$ then π' is unitarily equivalent to π_ϕ.*

Definition .5.33. The representation $\pi_{uni} : \bigoplus_{\phi\in S(\mathcal{A})} \pi_\phi$ is called the *universal representation* of $\mathcal{A}$.

Theorem .5.34 (Proposition 8.6.6. in [32])**.** *Supose that ϕ is a positive linear functional on $\mathcal{A}$. If $S \in \pi_\phi(\mathcal{A})'_w \cap [0, I]$, then $\phi_S := \langle S\pi_\phi(a)v_\phi \mid v_\phi\rangle$, $a \in \mathcal{A}$, defines a positive linear functional on $\mathcal{A}$ such that $\phi - \phi_S$ is positive. The mapping $S \to \phi_S$ is an order isomorphism of $\pi(\mathcal{A})'_w \cap [0, I]$ onto $[0, \phi]$, i.e., the map $S \to \phi_S$ is bijective, and $S_1 \leq S_2$ is equivalent to $\phi_{S_1} \leq \phi_{S_2}$, for arbitrary S_1, $S_2 \in \pi_\phi(\mathcal{A})' \cap [0, I]$.*

Remark .5.35. If $\mathcal{M}$ is an O^*-algebra on $D \subseteq H$, we have that $D_{\mathcal{M}} \subseteq H \subseteq D'_{\mathcal{M}}$, where $D'_{\mathcal{M}}$ is the dual space of $D_{\mathcal{M}}$.

Definition .5.36. Given a O^*-algebra $\mathcal{M}$ on D, we define de *weak commutant* $\mathcal{M}'_w$ of $\mathcal{M}$ as the set,

$$\mathcal{M}'_w = \{R \in B(H) \mid \forall S \in \mathcal{M} \forall v \in D, \langle RSv \mid v\rangle = \langle v \mid SRv\rangle\}$$

Definition .5.37. Given a O^*-algebra $\mathcal{M}$ on D, we define de *strong commutant* $\mathcal{M}'_w$ of $\mathcal{M}$ as the set,

$$\mathcal{M}'_s = \{R \in B(H) \mid \forall S \in \mathcal{M} \forall v \in D, RSv = SRv\}$$

Fact .5.38 (Proposition 7.2.10 in [32])**.** If $\pi : \mathcal{A} \to \Lambda(D)$ is closed and selfadjoint, $\pi(\mathcal{A})'_w = \pi(\mathcal{A})'_s$ and this set of operators is a von Neumann algebra. In this case any of these two sets are denoted by $\pi(\mathcal{A})'$ and called the *(bounded) commutant* of $\pi(\mathcal{A})$.

Notation .5.39. If $\pi : \mathcal{A} \to \Lambda(D)$ is closed and selfadjoint, we denote by $\pi(\mathcal{A})''$ the commutant (in the bounded operator sense) of $\pi(\mathcal{A})'$.

Definition .5.40.

- We define the *finite rank* algebra on D ($\mathcal{F}(D)$) in the following way:

  $$\mathcal{F}(D) := \{S \in \mathcal{L}^\dagger(D) \mid S \text{ has finite rank and } SH \subseteq D\}$$

- The closure of $\mathcal{F}(D)$ in the bounded topology on D is the *ideal of generalized compact operators* and is denoted by $\mathcal{V}(D)$.
- The *-algebra $\mathcal{Q}(D) : \mathcal{L}/\mathcal{V}(D)$ is called the *Generalized Calkin Algebra.*

Theorem .5.41 (Proposition 6.1.10 in [32])**.** *Suppose $\mathcal{M}$ is an O^* such that $\mathcal{D}_{\mathcal{M}}$ is a quasi-Frechet space. For each $S \in \mathcal{L}^+(D_{\mathcal{M}})$, the following three statements are equivalent:*

i) $S \in \mathcal{V}(D_{\mathcal{M}})$.

ii) S maps each weak null sequence in $\mathcal{D}$ into a null sequence sequence $\mathcal{D}_{\mathcal{M}}$.

iii) S maps every bounded set in $\mathcal{D}_{\mathcal{M}}$ into a relatively compact set in $\mathcal{D}_{\mathcal{M}}$.

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