Around Shelah's logic L^1_κ

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

CMU (Virtual) Model Theory Seminar - March 2021

Today and Next Thursday

Session I: Shelah's logic L^1_κ
- Logics "appropriate for model theory"
- Shelah's new logic
- Basic properties of L^1_κ
- Serious Properties of L^1_κ

Session II: approximation from below: $\mathsf{L}^{1,c}_\kappa$
- Variants: Approximations from above: chain logic, …

When is a logic "appropriate" for model theory?

- Of course, logics "similar to" $\mathsf{L}_{\omega,\omega}$, ${}^{\mathsf{cont}}\mathsf{L}_{\omega,\omega}$, …(they have Löwenheim-Skolem, Compactness, Interpolation, etc.)

When is a logic "appropriate" for model theory?

- Of course, logics "similar to" $\mathsf{L}_{\omega,\omega}$, ${}^{\mathsf{cont}}\mathsf{L}_{\omega,\omega}$, …(they have Löwenheim-Skolem, Compactness, Interpolation, etc.)
- $\mathsf{L}_{\omega_1,\omega}$? Compactness fails.

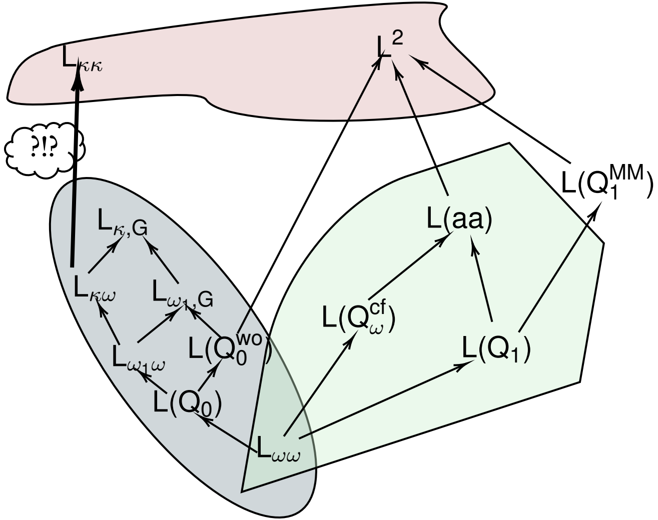
When is a logic "appropriate" for model theory?

- Of course, logics "similar to" $\mathsf{L}_{\omega,\omega}$, ${}^{\mathsf{cont}}\mathsf{L}_{\omega,\omega}$, ...(they have Löwenheim-Skolem, Compactness, Interpolation, etc.)
- $\mathsf{L}_{\omega_1,\omega}$? Compactness fails.
- $\mathsf{L}_{\kappa,\lambda}$...It depends strongly on κ (and λ)

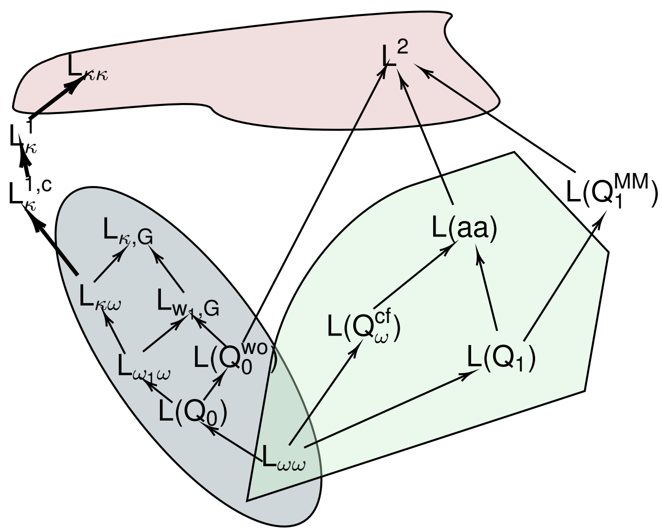
When is a logic "appropriate" for model theory?

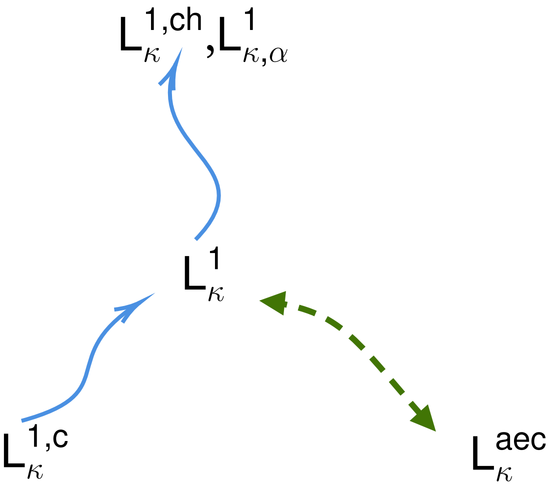
- Of course, logics "similar to" $\mathsf{L}_{\omega,\omega}$, ${}^{\mathsf{cont}}\mathsf{L}_{\omega,\omega}$, …(they have Löwenheim-Skolem, Compactness, Interpolation, etc.)
- $\mathsf{L}_{\omega_1,\omega}$? Compactness fails.
- $\mathsf{L}_{\kappa,\lambda}$…It depends strongly on κ (and λ)
- Väänänen: "infinitary logic may still serve as a 'yardstick' for model theoretic constructs, permits fragments of model theory and is preserved under (reasonable) forcing"…

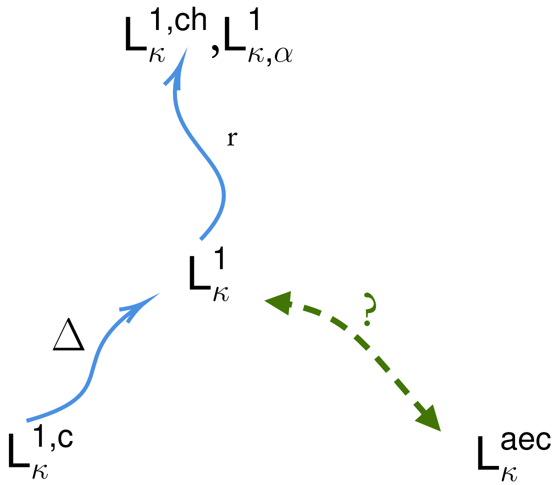
A (Väänänen) map of various infinitary logics



New Logics


$L_{\kappa\kappa}$
L^2
L^1_κ
$L^{1,c}_\kappa$
$L(Q_1^{MM})$
$L_{\kappa,G}$
$L(aa)$
$L_{\kappa\omega}$
$L_{w_1,G}$
$L(Q^{cf}_\omega)$
$L(Q_0^{wo})$
$L(Q_1)$
$L_{\omega_1\omega}$
$L(Q_0)$
$L_{\omega\omega}$


$L^{1,ch}_\kappa$,$L^1_{\kappa,\alpha}$
L^1_κ
$L^{1,c}_\kappa$
L^{aec}_κ


$L^{1,ch}_\kappa, L^1_{\kappa,\alpha}$
r
L^1_κ
Δ
?
$L^{1,c}_\kappa$
L^{aec}_κ

Interpolation

- Craig($L_{\kappa^+\omega}, L_{(2^\kappa)^+\kappa^+}$) (Malitz 1971).

Interpolation

- Craig($L_{\kappa^+\omega}, L_{(2^\kappa)^+\kappa^+}$) (Malitz 1971).
  If $\varphi \vdash \psi$, where φ is a τ_1-sentence and ψ is a τ_2-sentence and both are in $L_{\kappa^+\omega}$ then
  there exists $\chi \in L_{(2^\kappa)^+\kappa^+}(\tau_1 \cap \tau_2)$ such that
  $$\varphi \vdash \chi \vdash \psi.$$
- The original argument used "consistency properties". Other proofs have stressed the "Topological Separation" aspect of Interpolation.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

and $\mathsf{Craig}(\mathsf{L}^*)$.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

and $\mathsf{Craig}(\mathsf{L}^*)$.

- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

$$\bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\omega} \leq \mathsf{L}^1_\kappa \leq \bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\lambda^+}$$

and $\mathsf{Craig}(\mathsf{L}^1_\kappa)$.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$L_{\kappa^+\omega} \leq L^* \leq L_{(2^\kappa)^+\kappa^+}$$

and $\mathrm{Craig}(L^*)$.

- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

$$\bigcup_{\lambda<\kappa} L_{\lambda^+\omega} \leq L^1_\kappa \leq \bigcup_{\lambda<\kappa} L_{\lambda^+\lambda^+}$$

and $\mathrm{Craig}(L^1_\kappa)$.

- Moreover, in the case $\kappa = \beth_\kappa$, the logic L^1_κ also has a Lindström-type characterization as the maximal logic with a peculiar strong form of undefinability of well-order.

Lindström's Theorem

- A model theoretic characterization of first order logic.
- Maximal w.r.t. the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.

Decomposing "usual" proofs of Lindström's Theorem

- Build an EF-game, and its approximations.
- This gives also a model theoretic proof of Craig Interpolation.
- Craig's Theorem and Lindström's Theorem reveal parallel phenomena.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?
- We describe two partial answers, one approaching from below (Väänänen-V.), the other one from above (Džamonja, Väänänen).

SHELAH'S GAME $\ni^\beta_\theta(M, N)$.

ANTI	ISO
$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

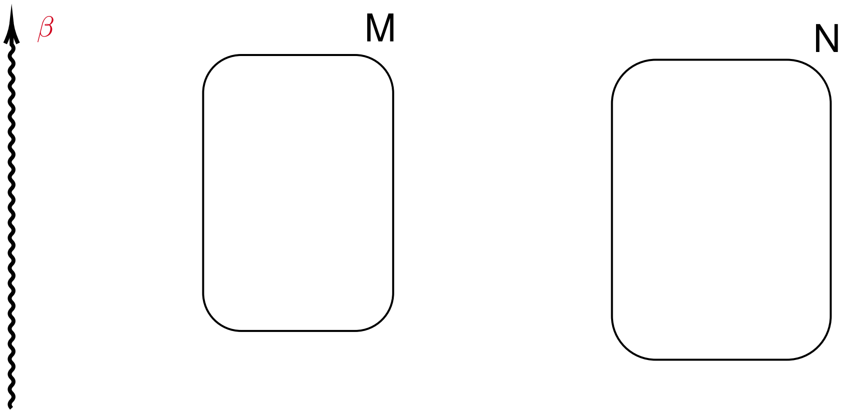
Constraints:

- $\mathrm{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2n}(m) \subseteq \mathrm{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(m) \subseteq \mathrm{ran}(g_{2n})$ for $m \leq n$.

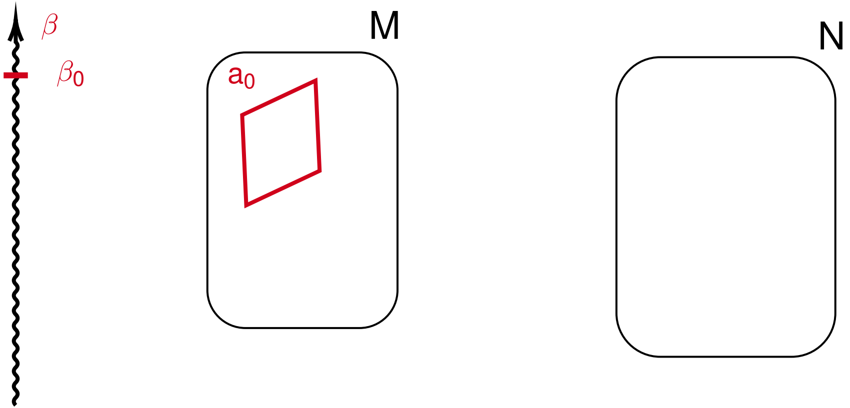
ISO wins if she can play all her moves, otherwise ANTI wins.

- $\mathsf{M} \sim^\beta_\theta \mathsf{N}$ iff ISO has a winning strategy in the game.
- $\mathsf{M} \equiv^\beta_\theta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim^\beta_\theta \mathsf{N}$.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv^\beta_\theta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L^1_κ.

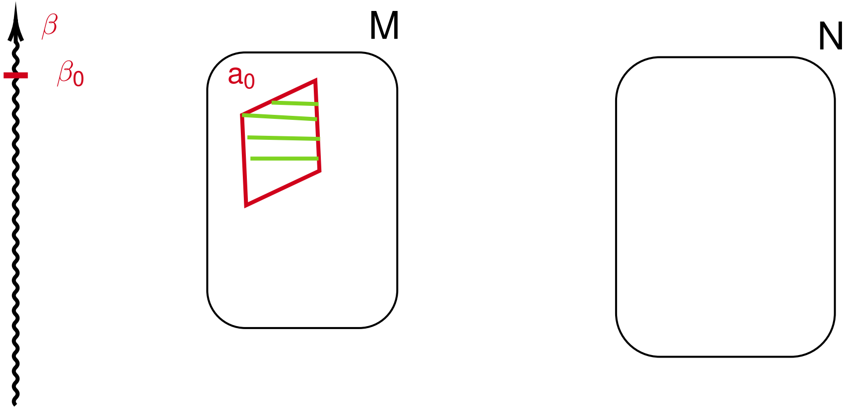
Shelah's game $\partial^\beta_\theta(M, N)$.



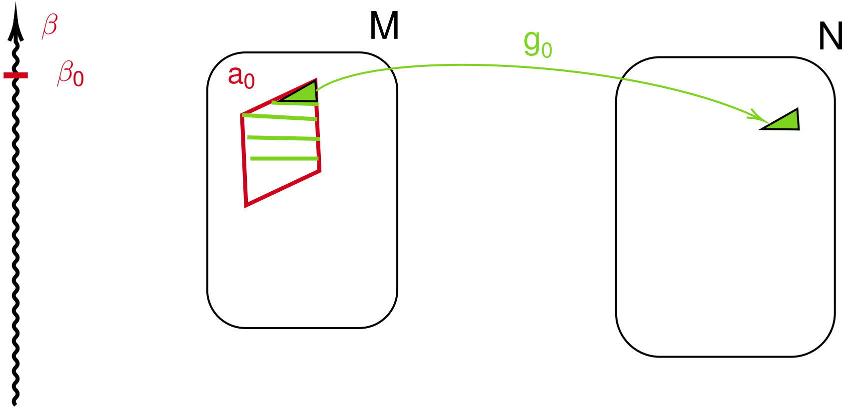
Shelah's game $\ni^\beta_\theta(M, N)$.



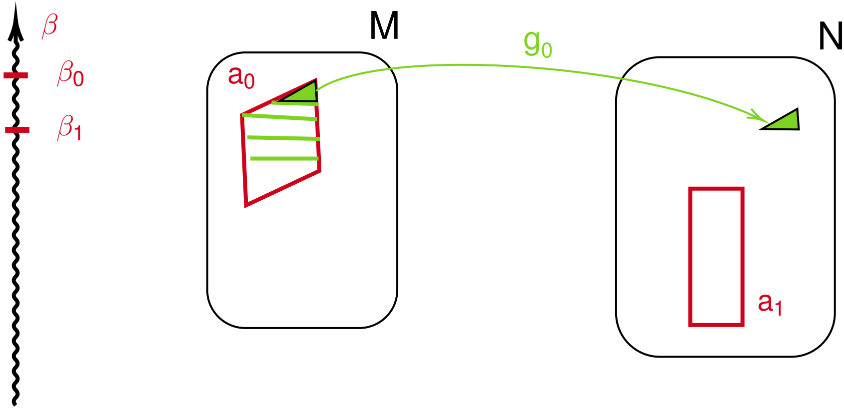
Shelah's game $\ni^\beta_\theta(M, N)$.



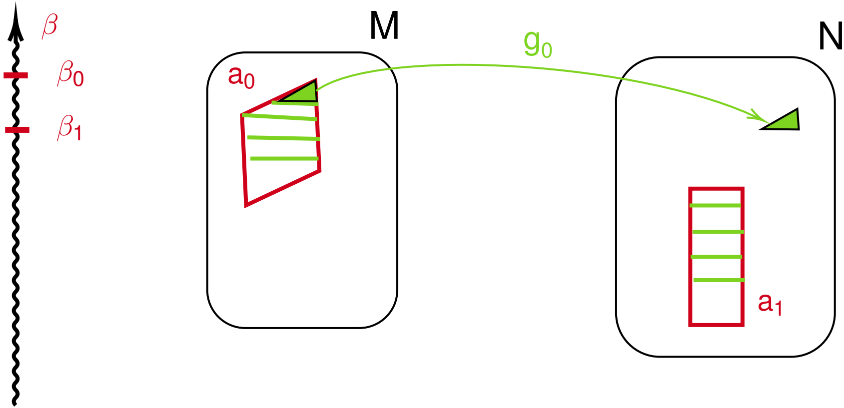
Shelah's game $\Game^\beta_\theta(M, N)$.



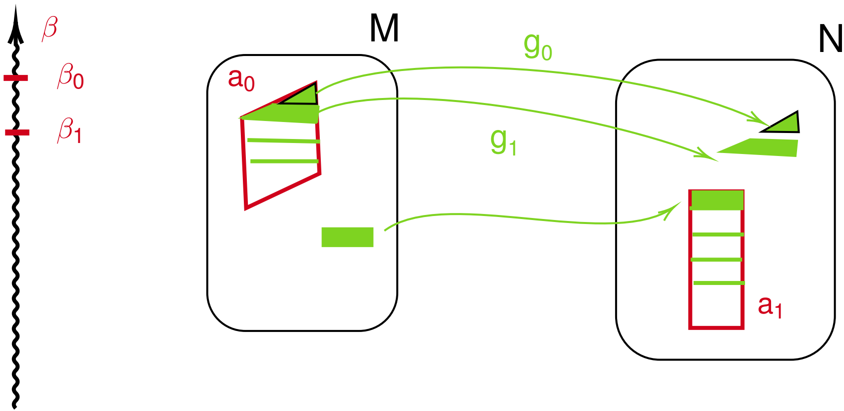
Shelah's game $\ni^\beta_\theta(M, N)$.



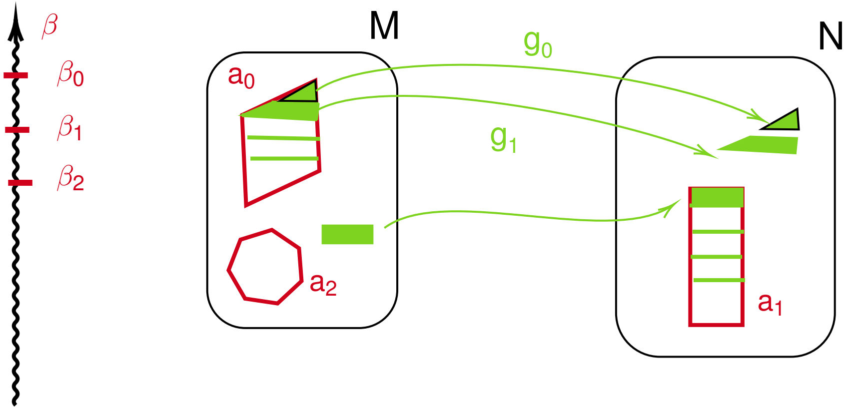
Shelah's game $\ni^\beta_\theta(M, N)$.



Shelah's game $\ni^\beta_\theta(M, N)$.



Shelah's game $\partial^\beta_\theta(M, N)$.



The definition of L^1_κ-sentences - again

- For M, N τ-structures, θ a cardinal, $\alpha \leq \theta$ an ordinal, $M \sim^\beta_\theta N$ iff ISO has a winning strategy in $\eth^\beta_\theta(M, N)$,
- $M \equiv^\beta_\theta N$ is defined as the transitive closure of $M \sim^\beta_\theta N$,
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv^\beta_\theta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L^1_κ.

Comparison with other logics: where is L^1_κ?

$$\bigcup_{\lambda<\theta} L_{\lambda^+,\omega} \leq L^1_{\leq\theta} \leq \bigcup_{\lambda<\beth_{\theta^+}} L_{\lambda^+,\lambda^+}$$

Key Lemma for second dominance:

$$M_1 \equiv_{L_{\beth_\beta(\theta)^+,\theta^+}} M_2 \ (\forall\beta < \theta) \quad \Longrightarrow \quad M_1 \sim_{\eth^{<\theta^+}_{\leq\theta}} M_2$$

(Induction on β: if **s** is a state in $\eth^{<\theta^+}_{\leq\theta}$, $\varphi(\bar{x})$ is a formula of $L_{\beth_\beta(\theta)^+,\theta^+}$ such that

$$M_1 \models \varphi[\mathrm{dom}(g_s)] \leftrightarrow M_2 \models \varphi[\mathrm{ran}(g_s)]$$

then **s** is a winning state for ISO in $\eth^{<\theta^+}_{\leq\theta}$.)

"Crucial Claim": closure under unions of ω-chains

Given $(M_n)_{n<\omega}$ a sequence of τ-structures and given $\psi(\bar{z}) \in L^1_{\leq\theta}(\tau)$, if

$$M_n \prec_{L_{\partial^+,\theta^+}} M_{n+1} \text{ , for all } n < \omega, \partial = \beth_{\theta^+}$$

"Crucial Claim": closure under unions of ω-chains

Given $(M_n)_{n<\omega}$ a sequence of τ-structures and given $\psi(\bar{z}) \in L^1_{\leq\theta}(\tau)$, if

$$M_n \prec_{L_{\partial^+,\theta^+}} M_{n+1} \text{ , for all } n < \omega, \partial = \beth_{\theta^+}$$

then

$$M_n \equiv_{L^1_\theta} M_\omega := \bigcup_{n<\omega} M_n$$

and

$$\forall \bar{a} \in {}^{lg(z)}M_0 \quad M_n \models \psi[\bar{a}] \Leftrightarrow M_\omega \models \psi[\bar{a}] \text{ for all } n < \omega.$$

(Weak) Downward Löwenheim-Skolem for L^1_κ

Assuming $\kappa = \beth_\kappa$,
for every sentence $\psi \in \mathsf{L}^1_\kappa$, if there exists M such that $\mathsf{M} \models \psi$ then there exists a model $\mathsf{N} \models \psi$, N of cardinality $< \kappa$.
for every $\psi \in \mathsf{L}^1_\kappa$ there is $\partial < \kappa$ such that: if N is a model of ψ of cardinality λ and $\mu = \mu^{<\partial}$ then some submodel M of N of cardinality μ is a model of ψ

The version of Undefinability of Well-Ordering

For this, the assumption $\kappa = \beth_\kappa$ seems crucial all along.
SUDWO (Strong Undefinability of Well Ordering):
If $\psi \in \mathsf{L}^1_\kappa(\tau)$, $|\tau| < \kappa$, $<$, R are binary predicates, $\mathsf{c}_1, \mathsf{c}_2$ constants from τ, THEN for every large enough $\mu_1 < \kappa$ for arbitrarily large $\mu_2 < \kappa$ we have:
if $\lambda > \mu_2$, $\mathfrak{A}$ is a τ-expansion of $(\mathsf{H}(\lambda), \in, \mu_1, \mu_2, <)$, with $<$ the order on ordinals, $\mathsf{R}^{\mathfrak{A}}$ being $\in$, $\mathsf{c}_1^{\mathfrak{A}} = \mu_1$, $\mathsf{c}_2^{\mathfrak{A}} = \mu_2 \ldots$ then there is $\mathfrak{B}$, $\mathsf{a}_\mathsf{n}, \mathsf{d}_\mathsf{n}$ $(\mathsf{n} < \omega)$ such that

- $\mathfrak{B} \models \psi \Leftrightarrow \mathfrak{A} \models \psi$,
- $\mathfrak{B} \models \mathsf{d}_{\mathsf{n}+1} < \mathsf{d}_\mathsf{n} < \mu_2$ for $\mathsf{n} < \omega$,
- $\mathfrak{B} \models \mathsf{a}_\mathsf{n} \subseteq \mathsf{a}_{\mathsf{n}+1}$ has cardinality $\leq \mu_1$,
- if $\mathsf{e} \in \mathfrak{B}$ and $\mathfrak{B} \models |\mathsf{e}| \leq \mu_1$ then $\mathfrak{B} \models \mathsf{e} \subseteq \mathsf{a}_\mathsf{n}$ for some n

The Lindström-like theorem

Let $\mathcal{L}$ be "a logic", let $\kappa = \beth_\kappa$. If $\mathcal{L}$ satisfies the following properties:

- $\mathcal{L}$ is nice (natural closure properties),
- the occurrence number of $\mathcal{L}$ is $\leq \kappa$,
- $\mathsf{L}_{\theta^+,\omega} \leq \mathcal{L}$, for $\theta < \kappa$,
- $\mathcal{L}$ satisfies SUDWO,

THEN
$\mathcal{L} \leq \mathsf{L}^1_\kappa$.

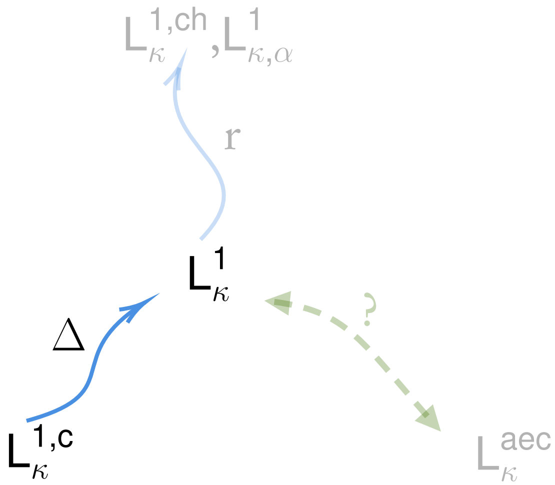
PLAN

Session I: Shelah's logic L^1_κ

- Logics "appropriate for model theory"
- Shelah's new logic
- Basic properties of L^1_κ
- Serious Properties of L^1_κ

Session II: approximation from below: $L^{1,c}_\kappa$

- Variants: Approximations from above: chain logic, …


$L^{1,ch}_\kappa$, $L^1_{\kappa,\alpha}$
r
L^1_κ
?
Δ
$L^{1,c}_\kappa$
L^{aec}_κ

Approaching L^1_κ from below (mod Δ)

- Joint work with **J. Väänänen**
- We define a sublogic $L^{1,c}_\kappa$ of L^1_κ ("Cartagena Logic") then the following is a formula of $L^{1,c}_\kappa$:

$$\forall \vec{x} \bigvee_f \bigwedge_n \phi_{f,n}(\vec{x}, \vec{y}).$$

- $L^{1,c}_\kappa$ has a recursive syntax.
- Many (but not all) of the nice properties of L^1_κ also hold for $L^{1,c}_\kappa$,
- The "distance" between the two logics is not large (Δ).

Syntax of $L^{1,c}_\kappa$

Suppose $2^\theta < \kappa$. The formulas of $L^{1,c}_{\kappa,\theta}$ are built from atomic formulas and their negations by means of the operation $\bigwedge_I, \bigvee_I$, where $|I| < \kappa$, and the following two operations:

Syntax of $L^{1,c}_\kappa$

Suppose $2^\theta < \kappa$. The formulas of $L^{1,c}_{\kappa,\theta}$ are built from atomic formulas and their negations by means of the operation $\bigwedge_I$, $\bigvee_I$, where $|I| < \kappa$, and the following two operations:
Suppose $\phi_A(\vec{x}, \vec{y})$, $A \subseteq \theta$, are formulas of $L^{1,c}_{\kappa,\theta}$ such that of the variables $\vec{x} = \langle x_\alpha : \alpha < \theta \rangle$ only those x_α for which $\alpha \in A$ occur free in $\phi_A(\vec{x}, \vec{y})$.

$$\forall \vec{x} \bigvee_f \bigwedge_n \phi_{f^{-1}(n)}(\vec{x}, \vec{y})$$

$$\exists \vec{x} \bigwedge_f \bigvee_n \phi_{f^{-1}(n)}(\vec{x}, \vec{y}),$$

where $\vec{x} = \langle x_\alpha : \alpha < \theta' \rangle$, $\theta' \leq \theta$ and $f : \theta' \to \omega$.

$$\forall \vec{x} \bigvee_f \bigwedge_n \phi_{f^{-1}(n)}(\vec{x}, \vec{y})$$

$$\exists \vec{x} \bigwedge_f \bigvee_n \phi_{f^{-1}(n)}(\vec{x}, \vec{y}),$$

where $\vec{x} = \langle x_\alpha : \alpha < \theta' \rangle$, $\theta' \leq \theta$ and $f : \theta' \to \omega$.

$$L^{1,c}_\kappa = \bigcup_{\theta<\kappa} L^{1,c}_{\kappa,\theta}$$

Subformulas of such formulas are the $\phi_A(\vec{x}, \vec{y})$, where $A \subseteq \theta'$. Thus the number of subformulas of such a formula is $2^{|\theta'|}$.

Cardinality quantifiers may be captured: $|P| < \theta$

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_f \bigwedge_n \Big(\bigwedge_{f(i)=n} P(x_i) \to \bigvee_{i \neq j \in f^{-1}(n)} (x_i = x_j) \Big)$$

says $|P| < \theta$.

An example of expressive power: no long chains

Example

Let $\theta < \kappa$ such that $\mathsf{cof}(\theta) > \omega$. Let $\mathsf{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_f \bigwedge_n \bigwedge_{i \neq j \in f^{-1}(n)} \neg x_i < x_j$$

says $<$ has no chains of length θ.

A covering property: the combinatorial core of L^1_κ!

The combinatorial core of Shelah's L^1_κ is captured by $L^{1,c}_\kappa$...

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$ and $\mathrm{len}(\vec{y}) = \omega$. The sentence

$$\forall \vec{x} \bigvee_{f} \bigwedge_{n} \exists \vec{y} \bigwedge_{g} \bigvee_{m} \bigwedge_{f(i)=n} \bigvee_{g(j)=m} R(y_j, x_i)$$

says every set of size $\leq \theta$ can be covered by countably many sets of the form $R(a, \cdot)$.

Corollary

Suppose $\theta < \kappa$. There is a sentence in $L^{1,c}_\kappa$ which has a model of cardinality θ if and only if $\theta^\omega = \theta$.

THE EF-GAME OF $L^{1,c}_\kappa$: $\Game^{\beta,c}_\theta(M,N)$.

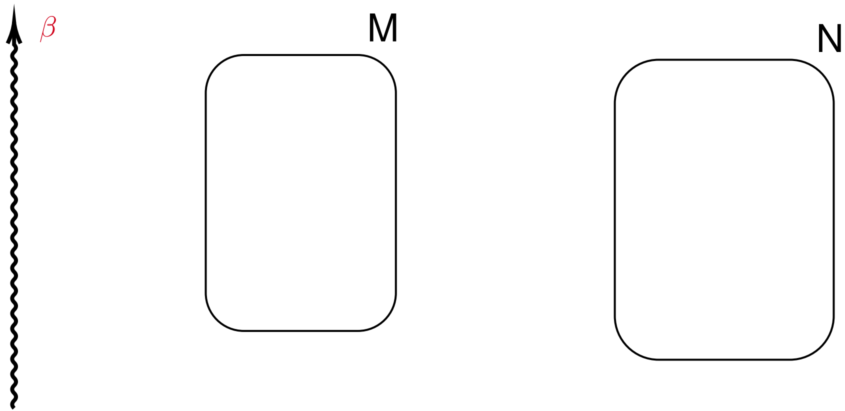
$\beta_0 < \beta$, $\vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$
$n_0 < \omega$	
	$g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0$, $\vec{a^1}$	
	$f_1 : \vec{a^1} \to \omega$,
$n_1 < \omega$	
	$g_1 : M \to N$ a p.i. $g_1 \supseteq g_0$
⋮	⋮

Constraints:

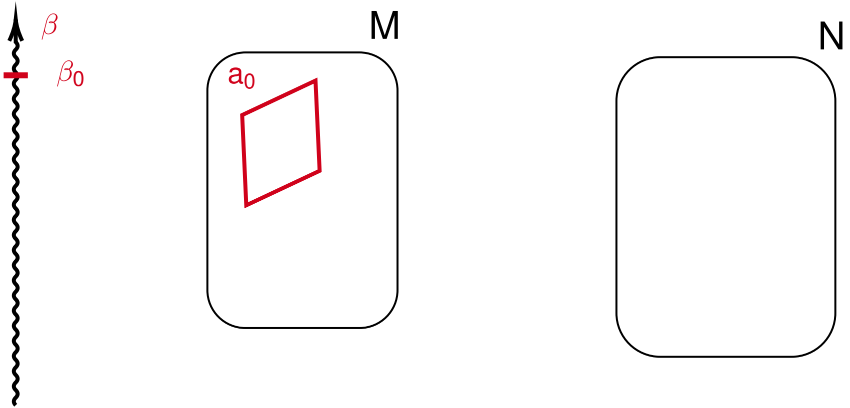
- $\mathrm{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2i}(n_{2i}) \subseteq \mathrm{dom}(g_{2i})$
- $f^{-1}_{2i+1}(n_{2i+1}) \subseteq \mathrm{ran}(g_{2i})$.

Player II wins if she can play all her moves, otherwise Player I wins.

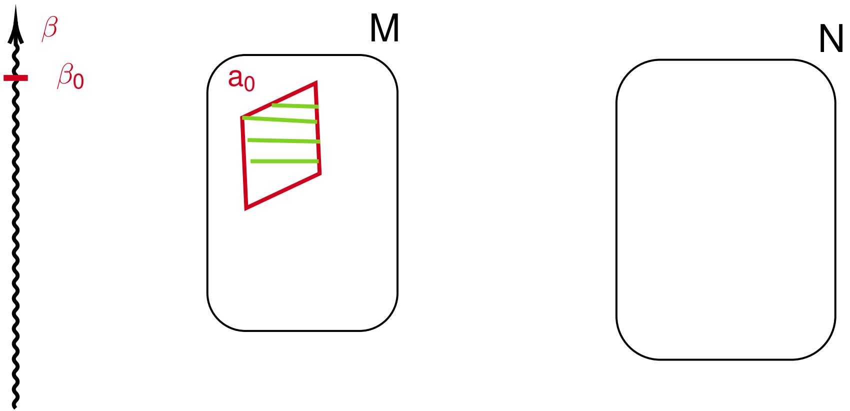
Our "Cartagena" game $\Game^{\beta,c}_\theta(M, N)$.



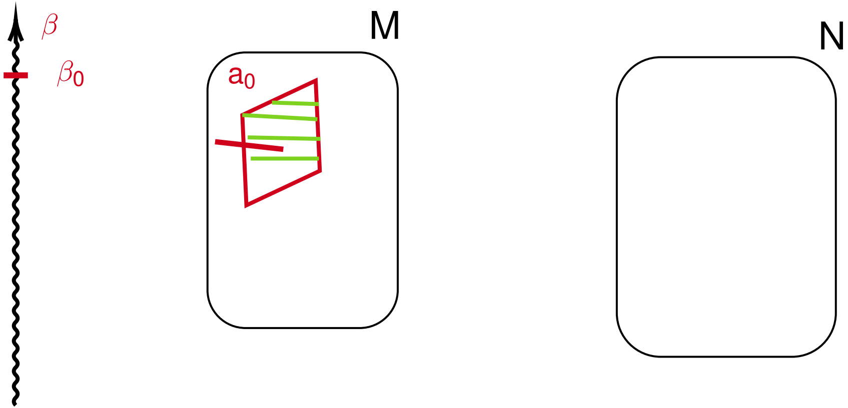
Our "Cartagena" game $\ni^{\beta,c}_\theta(M, N)$.



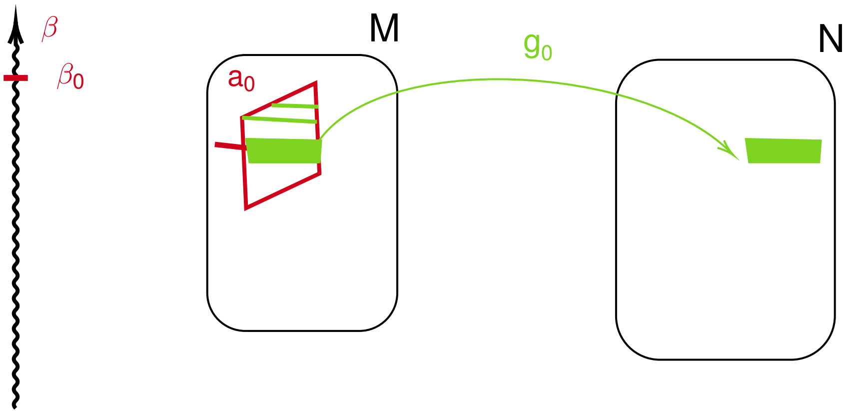
Our "Cartagena" game $\Game^{\beta,c}_\theta(M,N)$.



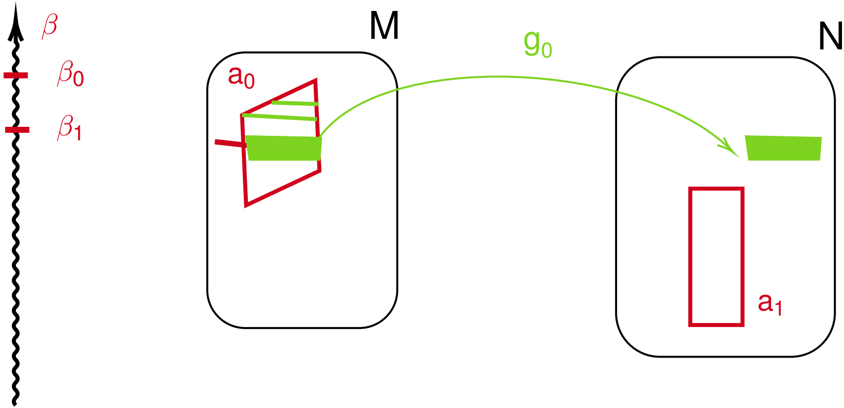
Our "Cartagena" game $\ni^{\beta,c}_\theta(M, N)$.



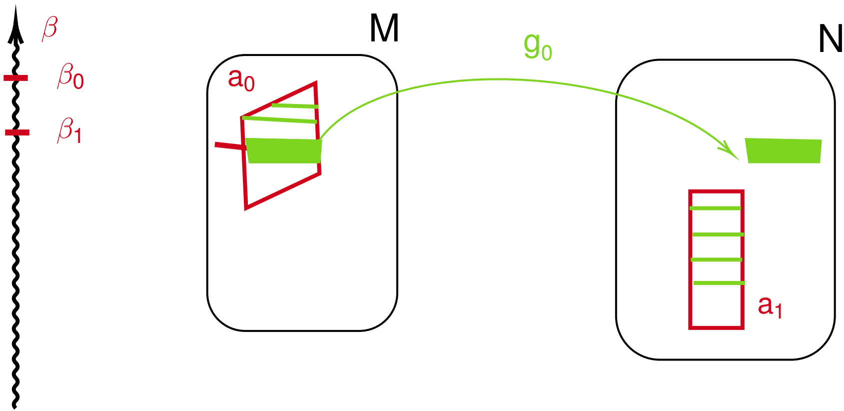
Our "Cartagena" game $\ni^{\beta,c}_\theta(M,N)$.



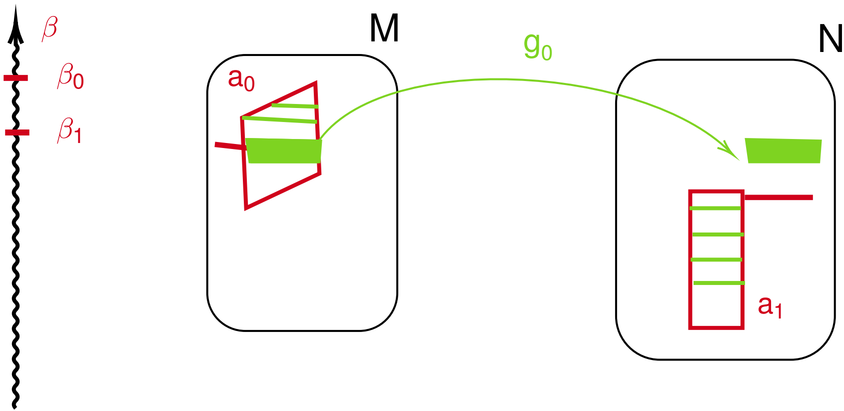
Our "Cartagena" game $\Game^{\beta,c}_\theta(M, N)$.



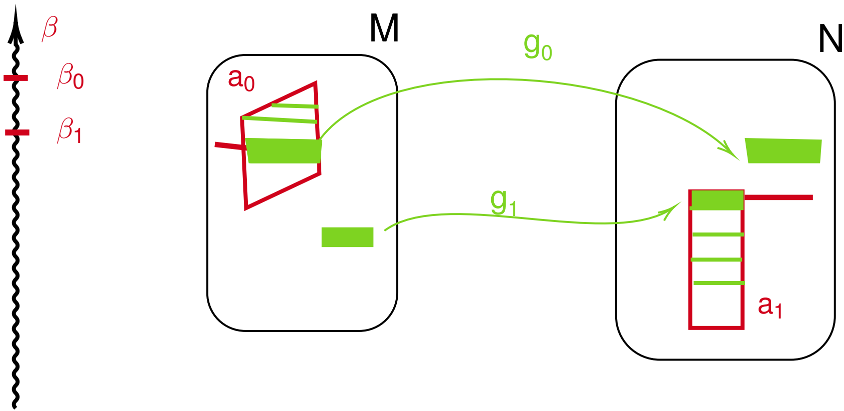
Our "Cartagena" game $\partial^{\beta,c}_\theta(M, N)$.



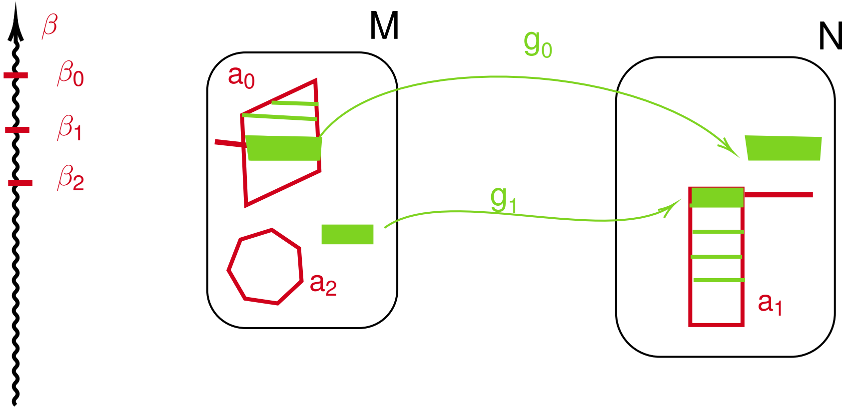
Our "Cartagena" game $\ni^{\beta,c}_\theta(M,N)$.



Our "Cartagena" game $\ni^{\beta,c}_\theta(M, N)$.



Our "Cartagena" game $\ni^{\beta,c}_\theta(M, N)$.



Theorem

The following are equivalent:

1. *Player II has a winning strategy in* $\Game^{\beta,\mathsf{c}}_\theta(\mathsf{M},\mathsf{N})$.
2. M *and* N *satisfy the same sentences of* $\mathsf{L}^{1,\mathsf{c}}_{\theta^+}$ *of quantifier rank* $\leq \beta$.

Corollary

$\mathsf{L}^{1,\mathsf{c}}_\kappa \leq \mathsf{L}^1_\kappa$.

Union Property of $L^{1,c}_\kappa$

Suppose Γ is a fragment of L^c_κ, i.e. a set of formulas closed under subformulas.
$M_n \prec_\Gamma M_{n+1}$ means that for formulas $\varphi(\bar{x})$ in Γ and $\bar{a} \in M_n$ we have

$$M_n \models \varphi(\bar{a}) \qquad \rightarrow \qquad M_{n+1} \models \varphi(\bar{a}).$$

Lemma (Union Lemma)

If $M_n \prec_\Gamma M_{n+1}$ *for all* $n < \omega$, *then* $M_n \prec_\Gamma M_\omega$ *where* $M_\omega = \bigcup_n M_n$.

Proof of the Union Lemma

Lemma (Union Lemma)

If $M_n \prec_\Gamma M_{n+1}$ *for all* $n < \omega$, *then* $M_n \prec_\Gamma M_\omega$ *where* $M_\omega = \bigcup_n M_n$.

Proof: Easy direction: $M_n \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies
$M_\omega \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.

Proof of the Union Lemma

Lemma (Union Lemma)

If $M_n \prec_\Gamma M_{n+1}$ *for all* $n < \omega$, *then* $M_n \prec_\Gamma M_\omega$ *where* $M_\omega = \bigcup_n M_n$.

Proof: Easy direction: $M_n \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies $M_\omega \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.
"Hard direction:" $M_n \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies $M_\omega \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.
So let $A \in [M_\omega]^\theta$, $\theta < \kappa$. **We treat** $A \cup M_m$ **separately** for each m.
Since $M_m \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$, there is $f_m : A \cap M_m \to \omega$ such that $M_m \models \bigwedge_n \varphi_{f_m^{-1}(n)}(A \cap M_m, \bar{a})$.

Proof of the Union Lemma

Lemma (Union Lemma)

If $M_n \prec_\Gamma M_{n+1}$ *for all* $n < \omega$, *then* $M_n \prec_\Gamma M_\omega$ *where* $M_\omega = \bigcup_n M_n$.

Proof: Easy direction: $M_n \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies $M_\omega \models \exists \bar{x} \bigwedge_f \bigvee_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.
"Hard direction:" $M_n \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies $M_\omega \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.
So let $A \in [M_\omega]^\theta$, $\theta < \kappa$. **We treat** $A \cup M_m$ **separately** for each m.
Since $M_m \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$, there is $f_m : A \cap M_m \to \omega$ such that $M_m \models \bigwedge_n \varphi_{f_m^{-1}(n)}(A \cap M_m, \bar{a})$. Let (e.g.) $f(a) = 2^m \cdot 3^{f_m(a)}$ for the smallest m such that $a \in M_m$. This f is the move of II. Then I plays m.

Claim

$M_\omega \models \varphi_{f^{-1}(m)}(A \cap f^{-1}(m), \bar{a})$.

But this follows from the Induction Hypothesis as $A \cap f^{-1}(m) = A \cap f_k^{-1}(m')$ for some m', k and $M_k \models \varphi_{f_k^{-1}(m')}(A \cap f_k^{-1}(m'), \bar{a})$. □

A consequence of the Union Lemma

Theorem
Assume $\kappa = \beth_\kappa$. *Then* $\Delta(L^{1,c}_\kappa) = L^1_\kappa$.

Further properties include

- LS theorems
- Undefinability of well order
- $\Delta(L^c_\kappa)$ contains any logic that satisfies the Union Lemma for $\prec_{\theta^+\theta^+}$, for arbitrary large $\theta < \kappa$. Shelah's L^1_κ is one such logic.

Note: Undefinability of well-order is a consequence of the LS property and the Union Lemma.

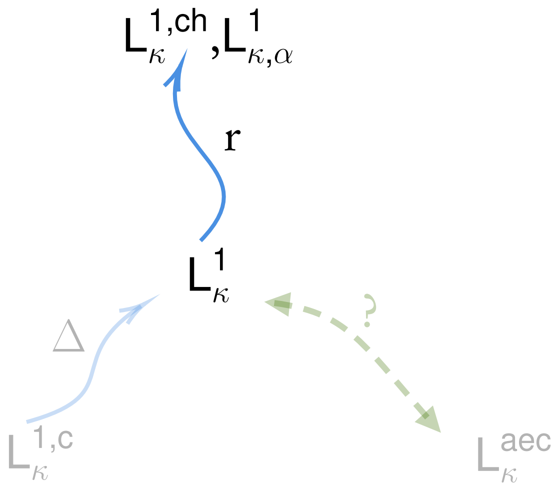
What is $\Delta(L)$?

- A model class $\mathcal{K}$ is $\Sigma(L)$ if it is the class of relativized reducts of an L-definable model class.
- A model class $\mathcal{K}$ is $\Delta(L)$ if both $\mathcal{K}$ and its complement are $\Sigma(L)$.
- $\Delta(L_{\omega\omega}) = L_{\omega\omega}$
- $\Delta(L_{\omega_1\omega}) = L_{\omega_1\omega}$
- $\Delta(\Delta(L)) = \Delta(L)$
- Δ preserves compactness, axiomatizability, Löwenheim-Skolem properties…

The advantages of $\mathsf{L}^{1,c}_\kappa$

- Simple syntax.
- Can express what L^1_κ does, at least implicitly.
- Its Δ-extension has Craig and Lindström Theorem.
- Undefinability of well-ordering is (also) a consequence of Caicedo's theorem on rigid structures and Uniform Reducibility of Pairs.

Musings on approximation from above



I: Chain logic $L^{1,ch}_\kappa$: Carol Karp

(This is recent work of Džamonja and Väänänen)

- Syntax: $L_{\kappa\kappa}$, κ singular strong limit of cof ω.
- Semantics in chain models ($M_0 \subseteq M_1 \subseteq \dots$)
- $\exists\vec{x}\phi$ means $\exists\vec{x}((\bigvee_n \bigwedge_j x_j \in M_n) \wedge \phi)$
- Craig($L^{1,ch}_\kappa$) (E. Cunningham, 1975)
- $L_{\kappa\omega} < L^{1,ch}_\kappa < L_{\kappa\kappa}$
- $L^1_\kappa \leq L^{1,c}_\kappa < L_{\kappa\kappa}$
- "Chu-transform" (Chu-spaces) is used as a device to compare logics.

II: From above, a new game (other splittings)

- L^1_κ is robust, but the lack of proper syntax if problematic.
- Väänänen and Velickovic define a deliberately stronger but simpler logic and then show that it is the same as L^1_κ, under conditions on κ.

The modified game $G^{1,\beta}_{\theta,\alpha}(M,N)$.

$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \alpha$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^0} \cup \vec{b^1} \to \alpha$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\mathrm{len}(\vec{a^n}) \leq \theta$, $\mathrm{len}(\vec{b^n}) \leq \theta$.
- $f_{i+1}(x) < f_i(x)$ if $f_i(x) \neq 0$.
- $f^{-1}_{2n}(0) \subseteq \mathrm{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(0) \subseteq \mathrm{ran}(g_{2n})$ for $m \leq n$.

Player II wins if she can play all her moves, otherwise Player I wins.

From above, the Väänänen-Velickovic variant of the game

- $G^{1,\beta}_{\theta,\alpha}(M, N)$ is the EF-game of a logic $L^1_{\theta,\alpha}$ up to the quantifier-rank β.
- If $\omega \leq \alpha \leq \alpha'$ and $\theta \leq \eta$, then $L^1_\theta \leq L^1_{\theta,\alpha} \leq L^1_{\theta,\alpha'} \leq L_{\eta^+\eta^+}$.
- If α is indecomposable, then "Player II has a winning strategy in $G^{1,\beta}_{\theta,\alpha}(M, N)$" is transitive and $L^1_{\kappa,\alpha}$ has a syntax (less clear than that of our $L^{1,c}_\kappa$).

From above, the Väänänen-Velickovic variant of the game

Theorem

If $\kappa = \beth_\kappa$ *and* α *is indecomposable, then* $\mathsf{L}^1_\kappa = \mathsf{L}^1_{\kappa,\alpha}$.

Comparison of the two games:

Trivially: If $\beta' \le \beta$, $\theta' \le \theta$ and $\alpha \le \alpha'$, then

$$\mathrm{II} \uparrow G^{1,\beta}_{\theta,\alpha}(A,B) \Rightarrow \mathrm{II} \uparrow G^{1,\beta'}_{\theta',\alpha'}(A,B).$$

Theorem

For every β there is β^* such that

$$\mathrm{II} \uparrow G^{1,\beta^*}_{2^\theta,\alpha}(A,B) \Rightarrow \mathrm{II} \uparrow G^{1,\beta}_{\theta,\omega}(A,B).$$

Here if $\kappa = \beth_\kappa$ and $\beta < \kappa$, then $\beta^* < \kappa$. The proof uses a lemma by Komjath and Shelah (A partition theorem for scattered order types. Combin. Probab. Comput. 12 (2003), no. 5-6, 621-626.)

For any α let $\mathsf{FS}(\alpha)$ be the tree of all descending sequences of elements of α. We use $\mathsf{len}(\mathsf{s})$ to denote the length of $\mathsf{s} \in \mathsf{FS}(\alpha)$.

Lemma (Komjath-Shelah 2003)

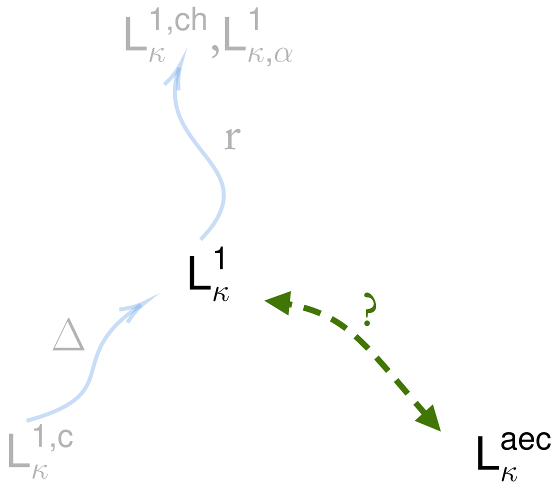
Assume that α *is an ordinal and* I *a set. Set* $\lambda = (|\alpha|^{|\mathsf{I}|^{\aleph_0}})^{++}$. *Suppose* $\mathsf{T} = \mathsf{FS}(\lambda)$ *and* $\mathsf{F} : \mathsf{T} \to \mathsf{I}$. *Then there is a subtree* $\mathsf{T}^* = \{(\delta^{\mathsf{s}}_0, \ldots, \delta^{\mathsf{s}}_{\mathsf{n}}) : \mathsf{s} = (\mathsf{s}_0, \ldots, \mathsf{s}_{\mathsf{n}}) \in \mathsf{FS}(\alpha)\}$ *of* T *and a function* $\mathsf{c} : \omega \to \mathsf{I}$ *such that for all* $\mathsf{s} \in \mathsf{T}^*$ *we have* $\mathsf{F}(\mathsf{s}) = \mathsf{c}(\mathsf{len}(\mathsf{n}))$.

Thank you!

For your attention!

References

- Mirna Džamonja and Jouko Väänänen, *Chain Logic and Shelah's Infinitary Logic*, ***ArXiV 1908.01177***, August 2019.
- Carol R. Karp, *Infinite-quantifier languages and ω-chains of models*, ***Proceedings of the Tarski Symposium*** (University of California, Berkeley, Calif., June 23–30, 1971), (William Craig, C. C. Chang, Leon Henkin, John Addison, Dana Scott, and Robert Vaught, editors), vol. XXV, American Mathematical Society, 1979, pp. 225–232.
- Saharon Shelah, *Nice Infinitary Logics*, ***Journal of the American Mathematical Society***, vol. 25 (2012), no. 2, pp. 395–427.
- Jouko Väänänen and Andrés Villaveces, *A syntactic approach to Shelah's logic* L^1_κ, ***pre-print***, 2020-2021.

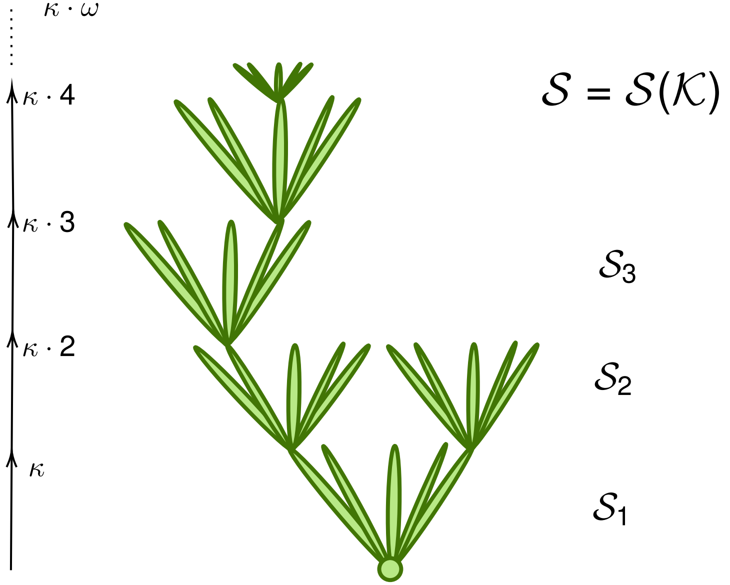

$L_\kappa^{1,ch}, L_{\kappa,\alpha}^1$
r
L_κ^1
?
△
$L_\kappa^{1,c}$
L_κ^{aec}

The canonical tree of an a.e.c.

This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_n := \{M \in \mathcal{K} \mid$ for some $\bar\alpha = \bar\alpha_M$ of length n, M has universe $\left\{a^*_\alpha \mid \alpha \in S_{\bar\alpha[M]}\right\}$ and $m < n \Rightarrow M \restriction S_{\bar\alpha\restriction m[M]} \prec_{\mathcal{K}} M\}$ (and $\mathcal{S}_0 = \left\{M_{empt}\right\}$),
- $\mathcal{S} = \mathcal{S}_{\mathcal{K}} := \bigcup_n \mathcal{S}_n$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalenty under $\subseteq$).

$\mathcal{S}(\mathcal{K})$



Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

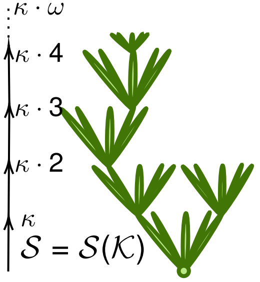
  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

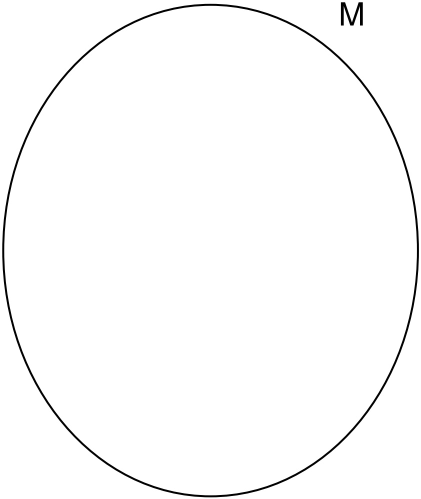
$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

Testing the class against the tree - Does M $\in \mathcal{K}$?





So we have <u>sentences</u> $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.
Let us take a closer look at low levels:

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?

The catch (beginnings)

When does $M \models \varphi_{1,0}$?
When in M,
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?
When in M,
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?
When in M,
$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

When does $\mathsf{M} \models \varphi_{2,0}$?

The catch (beginnings)

When does $M \models \varphi_{1,0}$?
When in M,
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

When does $M \models \varphi_{2,0}$?
When in M,
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

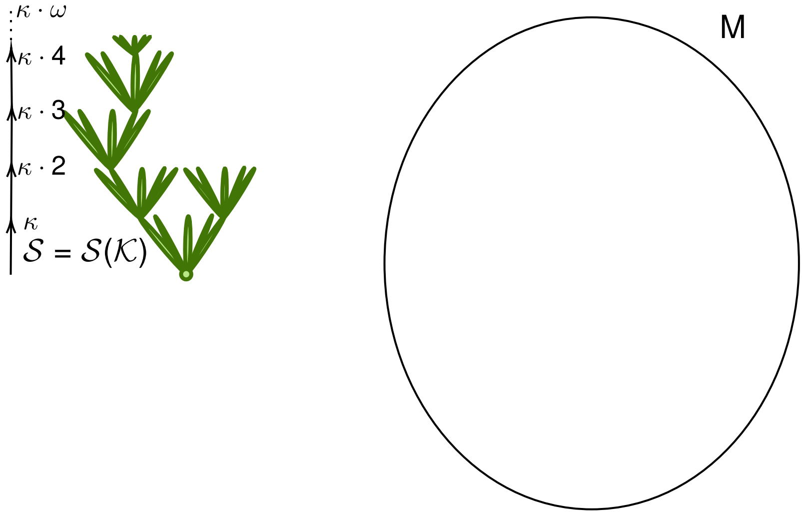
This is slightly more complicated to unravel:

$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

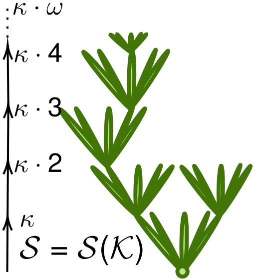
For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z…

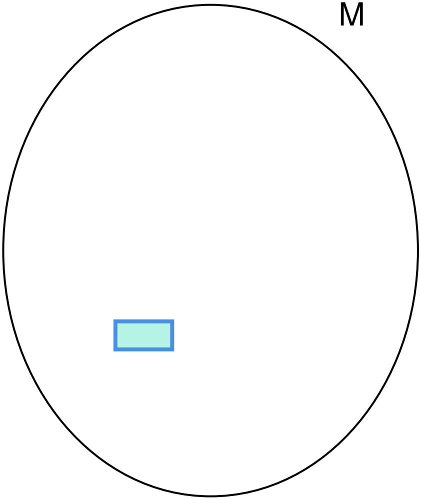
for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

The mezcal test - Does M $\in \mathcal{K}$?

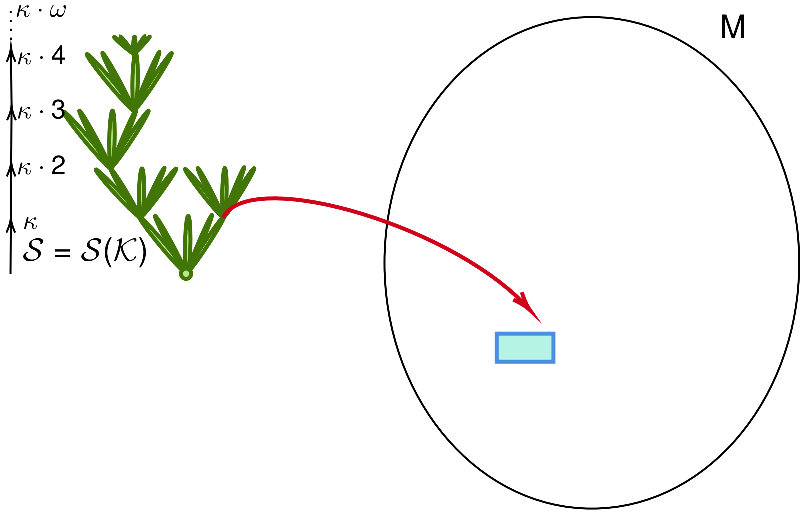


The mezcal test - Does M $\in \mathcal{K}$?

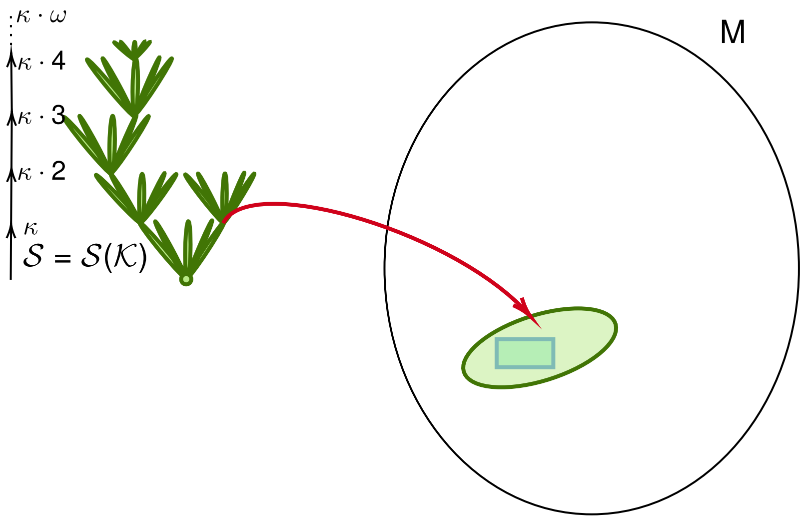




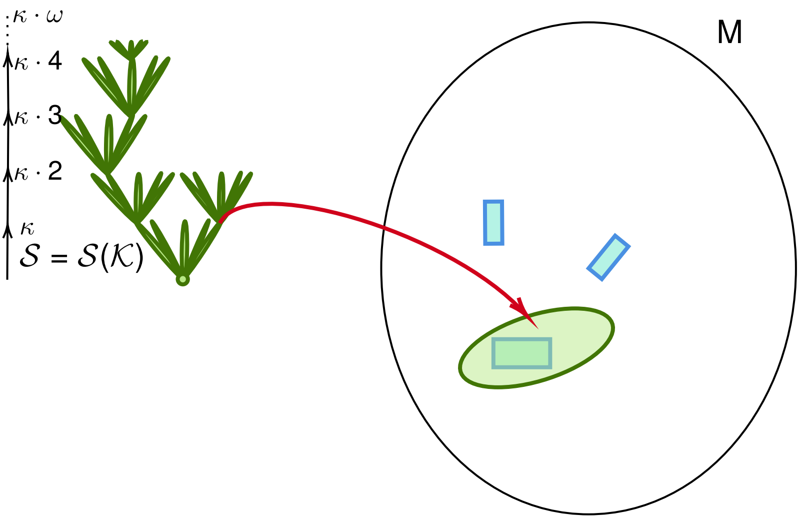
The mezcal test - Does M $\in \mathcal{K}$?



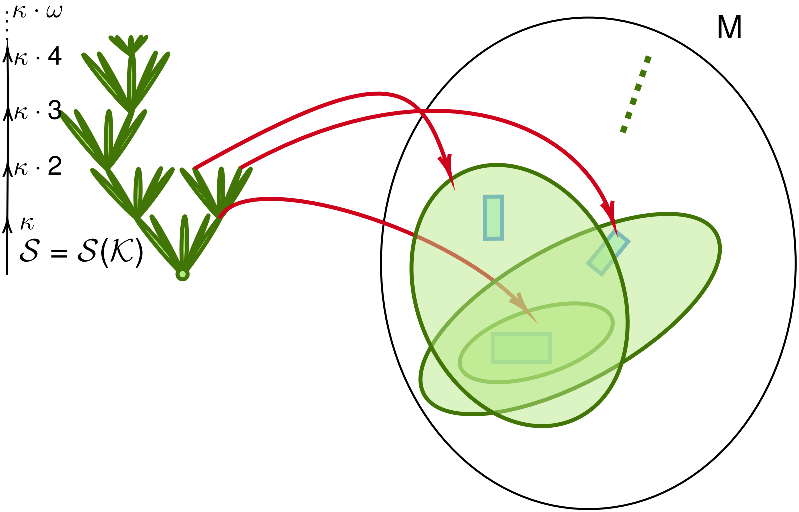
The mezcal test - Does M $\in \mathcal{K}$?



The mezcal test - Does M $\in \mathcal{K}$?



The mezcal test - Does M $\in \mathcal{K}$?



Theorem
$\mathsf{M} \in \mathcal{K}$ *implies* $\mathsf{M} \models \varphi_{\gamma,0}$ *for each* $\gamma < \lambda^+$

Theorem

$\mathsf{M} \models \varphi_{\beth_2(\kappa)^+ + 2, 0}$ *implies* $\mathsf{M} \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; the partition property due to Komjath and Shelah is the key…

The same partition property that worked for Väänänen and Velickovic's reduction of the game!

The tree property enables us to "reconstruct" M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$).
With this we can

- define "quantificational depth" of an aec (variants of Baldwin-Shelah (building on Mekler and Eklöf) give examples of high quantificational depth)…
- get definability of the "strong submodel relation" $\prec_{\mathcal{K}}$ …and genuine variants of a Tarski-Vaught test
- a grip on biinterpretability of AECs…