The Harris/Zilber theory of j: Understanding Types

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Mumford-Tate	Shimura curves, modular curves	Daw, Harris	Categoricity
	Moonshine uniformization	Cano, Plazas, V.	Categoricity

Classical j invariant

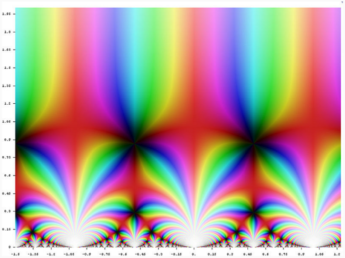
Klein defines the function (we call) "classical j"

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^3}$$

with g_2 and g_3 certain functions ("Eisenstein" series).



j-invariant on $\mathbb{C}$ (Wikipedia article on j-invariant)

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- $$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$$

  if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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- (Hilbert's 12th...)

j-covers and a path to categoricity

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Adam Harris provides a contrasting view of classical j invariants:

- An axiomatization in $L_{\omega_1,\omega}$ of j
- A convoluted proof of categoricity of this version of j
- Generalization of this analysis to higher dimensions (Shimura varieties).
- Analogies to pseudoexponentiation ("Zilber field") are strong, but the structure of j seems to have a much higher degree of complexity even than exp.

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$$\mathfrak{A} = \langle\langle H; \{g_i\}_{i\in\mathbb{N}}\rangle, \langle F, +, \cdot, 0, 1\rangle, j : H \to F\rangle$$

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where $\langle F, +, \cdot, 0, 1 \rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i \in \mathbb{N}} \rangle$ is a set together with countably many unary function symbols, and $j : H \to F$.

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Really, j is a **cover** from the action structure into the field $\mathbb{C}$.

The $L_{\omega_1,\omega}$-axiom - Crucial point: Standard fibers of the cover j

Let then

$$\mathrm{Th}_{\omega_1,\omega}(j) := \mathrm{Th}(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \to \bigvee_{i<\omega} x = \gamma_i(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1 \rangle, j : \mathbb{H} \to \mathbb{C})$.

This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits")

Categoricity of classical j

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($\mathcal{H}_i = (H_i, \{g^i_j\}_{j \in \mathbb{N}})$ and $\mathcal{F}_i = (F_i, +_i, \cdot_i, 0, 1)$)
there are isomorphisms φ_H, φ_F such that

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow[\varphi_H]{} & \mathcal{H}_2 \\ \Big\downarrow{\scriptstyle j_1} & & \Big\downarrow{\scriptstyle j_2} \\ \mathcal{F}_1 & \xrightarrow[\varphi_F]{} & \mathcal{F}_2 \end{array}$$

commutes.

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- Quasiminimal abstract elementary classes. These must be categorical (the model theory of these harks back to results of Shelah from the late 1980s - excellent classes, then combined with quasiminimal classes and much more dramatically simplified - in some cases - by Bays, Hart, Hyttinen, Kesälä and Kirby). [Linguistic closure, homogeneity, uniqueness of generic, CC, alg. control.]

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- On the way to the previous, reduction of types of elliptic curves to the torsion information, readable by limits of N-covers on the field structure. A quite strong form of QE.
- A theorem by Keisler on the number of types of categorical sentences of $\mathcal{L}_{\omega_1,\omega}$... (this will give a surprising twist).

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Arithmetic Geometry:

An instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

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- realizing the field type of a finite subset of a Hecke orbit over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in a finite subset ("Mumford-Tate" open image theorem used here) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E — the type of τ is determined by algebraic relations between torsion points of E.

Plan

Definability and Algebraicity

- In general, seeing that a compact Riemann surface is algebraic requires some form of Riemann-Roch. HOWEVER, Harris uses the general theory of covering spaces induced by inclusions of groups to realize their quotients as definable sets in the field sort.
- That reduction enables him (and later Daw and Zilber) to understand types as Galois representations.
- The map

$$p_N : \Gamma_N \setminus \mathbb{H} \to \mathbb{C}^{\psi(N)+1} : \tau \mapsto \Big(j(\tau), j(g_1\tau), \ldots, j(g_{\psi(N)\tau} \Big)$$

is biholomorphic onto its image, which is DEFINABLE in $(\mathbb{C}, +, \cdot, 0, 1)$.
$(\Gamma = PSL_2(\mathbb{Z}), \Gamma_N = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$\in \Gamma : b \equiv c \equiv 0, a \equiv d \bmod N\})$

Other definable quotients in the field sort

- Z_N, the image of $\mathbb{H}$ under p_N
- $q_N : Z_N \to Z_1 : \Big(j(\tau), j(g_1\tau), \ldots, j(g_{\psi(N)}\tau)\Big) \to j(\tau)$
- $p_g : \tau \to (j(g_1\tau), \ldots, j(g_n\tau)) \subseteq \mathbb{C}^n$
- When $M|N$, $q_{N,M} : Z_N \to Z_M$, the unique map taking $p_N(s_0)$ to $p_M(s_0)$, étale outside of the points of the branch locus $\{0, 1728\} \subset Z_1$ such that $q_N = q_M \circ q_{N,M} \ldots$

These help nail down the description of the types!

The types

If $\langle \mathsf{H}, \mathsf{F} \rangle \models \mathsf{Th}(\mathsf{j})$**, want a description of** $\mathsf{tp}_{\mathsf{H}}(\tau)$ **and** $\mathsf{tp}_{\mathsf{F}}(\mathsf{z})$**.**

The two-sorted type of a non-special tuple $\tau \subset \mathsf{H}$ **may be studied in the field sort.**

More specifically...

Since the algebraic curves $Z_g = \Gamma_g \setminus \mathbb{H}$ encode info on geometric interactions in $\langle H, F \rangle$, we may reduce to the field: (τ) is determined by

$$\bigcup_{g \subset G} {}_F(p_g(\tau)/\mathrm{dcl}(\emptyset) \cap F).$$

The key point is

$$\exists g \in G(g\tau_i = \tau_j) \qquad \underline{\text{iff}} \qquad \exists g \in G(j(\tau_i), j(\tau_j)) \in Z_g.$$

Later: the key info on types is contained in the Galois representations.

Building the model

Consider the "pro-étale cover"

$$\hat{\mathbb{C}} = \varprojlim_{g \subset G} Z_g.$$

(Morphisms: definable maps already discussed.) $\hat{\mathbb{C}}$ is pro-definable in $\langle \mathbb{C}, +, \cdot, \mathbb{Q}(j(S)) \rangle$; use $\hat{j} : \hat{\mathbb{C}} \to \mathbb{C}$.

The Galois action on $\hat{\mathbb{C}}$ is our tool.

$$\pi_1' := \varprojlim_{N} \mathrm{Aut}_{\mathrm{Fin}}(Z_N / Z_1).$$

Plan

The approach to categoricity

Types of independent tuples over the model-theoretic étale cover

$$\hat{\mathcal{U}} := \langle \hat{\mathsf{U}}, \mathsf{F} \rangle \xrightarrow{\hat{\mathsf{j}}} \langle \mathbb{C}, +, \cdot, \mathbb{Q}(\mathsf{j}(\mathsf{S})) \rangle$$

are the same as those in the standard model.

Categoricity will then be described in terms of Galois representation in the geometric étale fundamental group!

Keisler's Theorem, and its consequence in arithmetic geometry

Theorem (Keisler)
If an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ is $\aleph_1$-categorical then the set of complete m*-types realizable in models of ψ is at most countable.*

And a corollary

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Theorem

If $\mathsf{Th}^{\infty}_{\mathsf{SF}}(\mathsf{p})$ *is categorical,* L *a finite extension of* $\mathbb{Q}$, $\mathsf{K} = \mathsf{L}(\mathsf{j}(\mathsf{S}), \mathsf{x})$, $\mathsf{x} \in \mathbb{C}^{\mathsf{n}}$ *a strongly independent tuple, then the image of the homomorphism*

$$Aut(\mathbb{C}/\mathsf{K}) \longrightarrow \pi_1'^{\mathsf{n}}$$

has finite index.

Quoting from Eterović's paper[1]

Theorem 7.1 (see [15, Corollary 5.6]). *If an $\mathcal{L}_{\omega_1,\omega}$ sentence is $\aleph_1$-categorical, then the set of m-types over the empty set in models of this sentence is at most countable.*

Consider the inverse system over $S(\mathbb{C})$ of all those special subvarieties of the form $Z_{\overline{g}}$, where $\overline{g}$ is chosen so that $\Gamma_{\overline{g}}$ is normal in Γ, with the corresponding maps $\psi_{\overline{g},e} : Z_{\overline{g}} \to S(\mathbb{C})$. For every point $z \in S(\mathbb{C})$, the fibre over z of $\psi_{\overline{g},)}$ consists of finitely many points, and it carries a simply transitive action of $\Gamma/\Gamma_{\overline{g}}$ and a compatible action of $\mathrm{Aut}(\mathbb{C}/L)$, where L is a finitely generated field containing F_0 and the coordinates of z. If $p(x) = z$, then $\mathrm{tp}(x)$ knows in which of the $\mathrm{Aut}(\mathbb{C}/L)$-orbits in $\psi_{\overline{g},e}^{-1}(z)$ lies $p_{\overline{g}}(x)$.

The action of $\mathrm{Aut}(\mathbb{C}/L)$ may not be transitive on $\psi_{\overline{g},e}^{-1}(z)$, so by choosing different $\mathrm{Aut}(\mathbb{C}/L)$-orbits in the fibre, we can construct new types from $\mathrm{tp}(x)$. If the action $\mathrm{Aut}(\mathbb{C}/L)$ is never transitive (or if there are infintely many tuples $\overline{g}$ for which the action is not transitive), then we would be able to define uncountable many new 1-types. Keisler's theorem would then say that none of the theories we are looking at is $\aleph_1$-categorical.

However, if a Galois representation $\rho_{\tilde{y}} : \mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma}$ (following the notation of §3.5) has open image, then the number of tuples for which the action of $\mathrm{Aut}(\mathbb{C}/L)$ is non-transitive, is finite.

Now, it does not make sense to expect $\rho_{\tilde{y}}$ to have open image unless z is Hodge-generic. If $\mathrm{spcl}(z) = V \subsetneq S(\mathbb{C})$, then we need to "relativise" the inverse system of varieties $Z_{\overline{g}}$ to V, meaning that we need to consider the inverse system of varieties of the form $Z_{\overline{g}}^{V,i}$ (where we choose an index i and fix it), and so the sensible condition to ask in this case is that a representation $\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma^{V,i}}$ have open image.

[1] Categoricity of Shimura Varieties. Submitted. ArXiv version, 2019.

Getting categoricity: the role of quasiminimality

Keisler's theorem is used then to show that categoricity (or really, few types) implies the condition above (finite index).

The other direction uses quasiminimal classes.

The closure operator is $\mathrm{cl} := \mathrm{j}^{-1} \circ \mathrm{acl} \circ \mathrm{j}$.

A crucial lemma shows $\aleph_0$-homogeneity over $\mathrm{dcl}(\emptyset)$, from the finite index condition mentioned earlier. The key point is that the finite index condition implies that all the info in $(\mathrm{h}\tau)/\mathrm{L}$ is contained in the field type of a finite subset of the Hecke orbit. This has strong similarities to earlier uses of the so-called "thumbtack lemma" in other situations.

Ideas/Translations/Questions to the geometers

1. Original context: Galois representation on the Tate module of an abelian variety A (limit of torsion points). Conjecturally, the image of such a Galois representation, which is an ℓ-adic Lie group for a given prime number ℓ, is determined by the corresponding Mumford–Tate group G (knowledge of G determines the Lie algebra of the Galois image).
2. Unfolding categoricity through the geometry seems to be the main question at this point - one that the Zilber school has pushed quite far but is still in its infancy.
3. Connection to properties of extendability of local sections to global sections (in sheaf cohomology)

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Same picture, much more general

Generalizing a bit the previous (but the picture is the same):

- S a modular curve: $\mathbb{H}/\Gamma$ where Γ is a "congruence subgroup" of $GL_2(\mathbb{Q})$,

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- S a modular curve: $\mathbb{H}/\Gamma$ where Γ is a "congruence subgroup" of $GL_2(\mathbb{Q})$,
- X^+ a set with an action of $G^{ad}(\mathbb{Q})^+$,
- $p : X^+ \to S(\mathbb{C})$ satisfies
  - (SF) Standard fibers,
  - (SP) Special points,
  - (M) Modularity.

Same picture, much more general

If any other map $q : X^+ \to S(\mathbb{C})$ also satisfies SF, SP and M, then there exist a $G^{ad}(\mathbb{Q})^+$-equivariant bijection φ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing the field of definition of S such that

$$\begin{array}{ccc} X^+ & \xrightarrow{\varphi} & X^+ \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ S(\mathbb{C}) & \xrightarrow{\sigma} & S(\mathbb{C}) \end{array}$$

Eterović's Generalization

In 2018, Eterović provided a tighter control on the language used for Shimura varieties, that enabled him to treat in a more robust way not just categoricity issues around Shimura varieties, but linking them in a more clear way with the Ax-Schanuel situation central to the differential treatment of the subject. I will not get into the details of this language change, but will just mention two things:

- Three languages, three theories of (covers of) Shimura varieties (inverse limit constructions):

$$\mathrm{Th}(\mathbb{p}) \qquad \tilde{\mathrm{T}}(\hat{\mathbb{p}}) \qquad \tilde{\mathrm{T}}(\tilde{\mathbb{p}}).$$

- The same framework as in Harris/Daw-Zilber for categoricity.

Plan

Comparing two/three situations

I follow Scanlon's lecture for the Bogotá Seminar here:

The three approaches to the model theory of covers:

- Harris (and Zilber, Eterović, Daw): use of Schanuel-like descriptions, categoricity in $L_{\omega_1,\omega}$.
- o-minimal complex analysis (Peterzil-Starchenko): definability of certain covers on fundamental domains (generalizing definability of period maps; Klinger/Orr),
- "inverting the covering"

$$q^{-1} : \Gamma \setminus D \to D \hookrightarrow D^\vee$$

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Comparing these three? (To be continued. . .)