SOBRE DAW-HARRIS (CATEGORICIDAD Y FUNCIÓN-J)

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Contents

La primera sesión larga del seminario tiene como tema la pregunta:

¿qué entendemos en este momento, y (tal vez más interesante) qué no entendemos, del trabajo de la tesis de Harris con Zilber y Pila - y luego su trabajo con Daw[1]?

Dado que tenemos tres enfoques distintos en el seminario (Cano, Plazas, V.), habrá tres respuestas a la misma. Planeo enfatizar aspectos modelo-teóricos de la prueba, hacer preguntas sobre aspectos de representación de Galois (me ha servido leer sobre el tema a R. Taylor [3]) y sobre posibles direcciones a futuro.

1. El teorema - conceptos básicos

An $L_{\omega_1,\omega}$ axiomatization of j: Let L be a language for two-sorted structures of the form

$$\mathfrak{A} = \langle \langle H; \{g_i\}_{i \in \mathbb{N}} \rangle, \langle F, +, \cdot, 0, 1 \rangle, j : H \to F \rangle$$

where $\langle F, +, \cdot, 0, 1\rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i\in\mathbb{N}}\rangle$ is a set together with countably many unary function symbols, and $j : H \to F$. Really, j is a **cover** from the *action* structure into the field $\mathbb{C}$.

Let then

$$\mathrm{Th}_{\omega_1,\omega}(j) := \mathrm{Th}(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \to \bigvee_{i<\omega} x = \gamma_i(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{H}, +, \cdot, 0, 1\rangle, j : \mathbb{H} \to \mathbb{C})$.

This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits").

Theorem 1. *(Harris, assuming Mumford-Tate Conj.)*

The theory $\mathrm{Th}_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geqslant \aleph_0$ *is categorical in all infinite cardinalities. I.e., given two models* $M_1 = (\mathcal{H}_1, F_1, j_1 : H_1 \to F_1)$ *and* $M_2 = (\mathcal{H}_2, F_2, j_2 : H_2 \to F_2)$ *of the same infinite cardinality* ($\mathcal{H}_i = (H_i, \{g^i_j\}_{j\in\mathbb{N}})$ *and* $\mathcal{F}_i = (F_i, +_i, \cdot_i, 0, 1)$) *there are isomorphisms* φ_H, φ_F *such that*

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\varphi_{\mathcal{H}}} & \mathcal{H}_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ \mathcal{F}_1 & \xrightarrow[\varphi_{\mathcal{F}}]{} & \mathcal{F}_2 \end{array}$$

commutes.

In his proof, A. Harris uses an instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

- Identifying $\mathrm{dcl}^M(\emptyset)$ with $\mathrm{dcl}^{M'}(\emptyset)$ to start the back-and-forth argument.
- Assume we have $\langle\bar{x}\rangle \approx \langle\bar{x}'\rangle$ and take new $y \in M$ – we need to find $y' \in M'$ to extend the partial isomorphism (satisfying the same quantifier free type)
- realizing the field type of a finite subset of a *Hecke orbit* over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in a finite subset ("Mumford-Tate" open image theorem used here) ... every point $\tau \in \mathcal{H}$ corresponds to an elliptic curve E — the type of τ is determined by algebraic *relations between torsion points* of E.

1.1. j-like mappings on modular curves.

Generalizing a bit the previous (but the picture is the same):

let S be modular curve: $\mathcal{H}/\Gamma$ where Γ is a "congruence subgroup" of $GL_2(\mathbb{Q})$, X^+ a set with an action of $G^{ad}(\mathbb{Q})$, $p : X^+ \to S(\mathbb{C})$ satisfies

- (SF) Standard fibers,
- (SP) Special points,
- (M) Modularity.

If any other map $q : X^+ \to S(\mathbb{C})$ also satisfies SF, SP and M, then there exist a $G^{ad}(\mathbb{Q})^+$-equivariant bijection φ and $\sigma \in Aut(\mathbb{C})$ fixing the field of definition of S such that

$$\begin{array}{ccc} X^+ & \xrightarrow{\varphi} & X^+ \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S(\mathbb{C}) & \xrightarrow[\sigma]{} & S(\mathbb{C}) \end{array}$$

1.2. Ideas, questions to the geometers.

(1) Modularity Axioms ("Hrushovski predimension" style conditions) in $Th(D, q, S)$):
   - $MOD^1_{\bar{g}} := \forall x \in D(q(g_1x), \cdots, q(g_nx)) \in Z_{\bar{g}}$,
   - $MOD^2_{\bar{g}} := \forall z \in Z_{\bar{g}} \exists x \in D(q(g_1x), \cdots, q(g_nx)) \in Z_{\bar{g}}$.

(2) Other axioms control "special points" (unique fixed points by the action of some element) and "generic points" (fixed by no element of the group $G^{ad}(\mathbb{Q})$).

(3) A theorem of Keisler on the number of types realized in models of size $\aleph_1$ of sentences in $L_{\omega_1,\omega}$ has the following consequence: *uncountable categoricity implies the geometric condition [Mumford-Tate].*

(4) Mumford-Tate: given A an abelian variety of dimension g defined over a field K, and $\rho : G_K \to Aut(T(A))$ the image of $Gal(\bar{K}/K)$ is open.

(5) Original context: Galois representation on the Tate module of an abelian variety A (limit of torsion points). Conjecturally, the image of such a Galois representation, which is an ℓ-adic Lie group for a given prime number ℓ, is determined by the corresponding Mumford-Tate group G (knowledge of G determines the Lie algebra of the Galois image).

(6) Unfolding categoricity through the geometry seems to be the main question at this point - one that the Zilber school (here present!) has pushed quite far.

(7) Connection to properties of extendability of local sections to global sections (in sheaf cohomology).

2. The rôle of Mumford-Tate (Jorge Plazas)

Plazas describes the construction of the representation, that will be used in the proof of quantifier elimination.

2.1. Special points are "trivial". Harris proved that the $L_{\omega_1,\omega}$-theory of the structure

$$\langle \mathfrak{H}, j, \mathbb{C}, \mathbb{Q}(j(S)) \rangle$$

is categorical. The canonical model is a model in the language consisting of:

- unary symbols $(g_i)_{i<\omega}$ for the action of $G = SL_2(\mathbb{Q})/\mathbb{Q}^*$ over the set $\mathfrak{H}$,
- the field structure of $\mathbb{C}$,
- constants for elements in $\mathbb{Q}(j(S))$ where S is the set of *special points*
- the j-mapping

A point s of $\mathfrak{H}$ is **special** if there exists a nontrivial $\gamma \in G$ such that $\gamma s = s$. Notice that if s is special then $[\mathbb{Q}(s) : \mathbb{Q}] = 2$. This is clear: if $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\frac{as+b}{cs+d} = s$, i.e.., $cs^2 + (d-a)s - b = 0$, so s is the root of a quadratic polynomial. Moreover, if s is special, then $j(s)$ is algebraic over $\mathbb{Q}(s)$ (this is a nontrivial theorem). Also, if $k = \mathbb{Q}(s)$ then $k(j(s))$ is the maximum unramified extension of k. Finally, s is special if and only if both $s, j(s)$ are algebraic.

Let $K := \mathbb{Q}(j(S)) = \mathbb{Q}(\{j(s) \mid s \text{ is special}\})$. By the previous remarks, this clearly is an algebraic extension (of infinite degree of course) of $k = \mathbb{Q}(S) = \mathbb{Q}(\{s \mid s \text{ is special}\})$, and a fortiori algebraic over $\mathbb{Q}$.

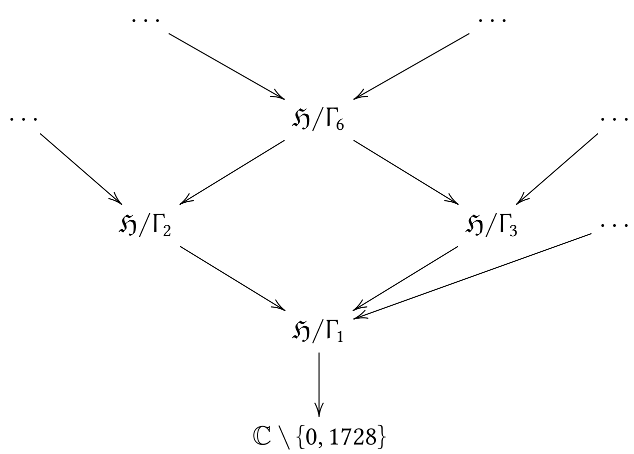
This shows the essential *triviality* of special points: $\mathbb{Q}(j(S))$ is sufficiently rigid so as to be codifiable by something profinite on top of $\mathbb{Q}(j(S))$: the "modular tower".

2.2. **The modular tower.** Fix $N > 0$. Let

$$\Gamma(N) = \Gamma_N = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid \gamma \equiv d \mod N \right\}.$$

The congruence is computed componentwise. So for instance $\Gamma_1 = SL_2(\mathbb{Z})$, and if $N \mid M$ then $\Gamma_M < \Gamma_N$. Also, $\mathfrak{H}/\Gamma_1$ is unramified everywhere, except at $i, e^{\frac{2\pi i}{6}}$; also, $\mathbb{C}\setminus\{0, 1728\} = \mathbb{A}^1(\mathbb{C})$.

We consider at the same time all the quotients of $\mathfrak{H}$ by the groups Γ_N - we clearly have the following diagram:



Fix $\Gamma = \Gamma_N$ or some subgroup of $SL_2(\mathbb{Q})$ contained in *some* Γ_N. Then $\mathfrak{H}/\Gamma$ is a Riemann surface. We may embed $\mathfrak{H}/\Gamma$ (as an affine variety over $\mathbb{C}$) into $\mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}/\Gamma$ (as a projective variety over $\mathbb{C}$). Fix the notation $Z_N := \mathfrak{H}/\Gamma_N$.

3. Problemas de eliminación de cuantificadores

3.1. **Realising finite pieces of types - QE, completeness.** Fix a Shimura variety S (think elliptic curve) and let $\mathbf{p}$ be the corresponding two-sorted structure. Now let

$$\mathbf{q} = \langle \mathbf{D}, \mathbf{S}, q \rangle, \mathbf{q}' = \langle \mathbf{D}', \mathbf{S}', q' \rangle$$

be models of $Th(\mathbf{p})$.

The following proposition seems to be crucial:

Proposition 2. *Let $x \in D$, $\overline{g} = (g_1, \ldots, g_n)$ a tuple of elements of $G^{ad}(\mathbb{Q})^+$, L a subfield of F containing $E^{ab}(\Sigma)$. Suppose σ is an embedding of L into F' fixing $E^{ab}(\Sigma)$. Then there exists an element $x' \in D'$ such that*

$$(q'(g_1 x'), \ldots, q'(g_n x')) \in S(F')^n$$

is a realization of

$$qftp_{\mathcal{L}_S}\left((q(g_1 x), \ldots, q(g_n x))/L\right)^{\sigma}.$$

The proof of this...

Proposition 3. *Hypotheses as before... If*

$$\mathbf{q} = \langle \mathbf{D}, \mathbf{S}, q \rangle, \mathbf{q}' = \langle \mathbf{D}', \mathbf{S}', q' \rangle$$

are ω*-saturated models of* $\mathrm{Th}(\mathbf{p})$ *and*

$$\rho : D \cup S(F) \to D' \cup S(F')$$

is a partial isomorphism with finitely generated domain U*, then given any* $\alpha \in D \cup S(F)$*,* ρ *extends to the structure* $\langle U \cup \{\alpha\} \rangle$*.*

Notes on the proof: there are many things to be yet clarified but...

- $U = U_D \cup U_S$ - the D-part, the $S(F)$-part. The D-part is the union of the $G^{ad}(\mathbb{Q})^+$-orbits of *finitely* many $x \in D$ and U_S is $S(L)$ for some field L generated by the coordinates of the images of these orbits in $S(F)$ along with finitely many other points in $S(F)$.
- ρ is a $G^{ad}(\mathbb{Q})^+$-equivariant injection $\varphi : U_D \to D'$ and an embedding $S(L) \to S(F')$ induced by an embedding $\sigma : L \to F'$ - *fixing* $E^{ab}(\Sigma)$.
- Case 1: $\alpha = z \in S(F)$. WLOG $z \notin q(U_D)$ - otherwise $z \in U_S$. Now remember than $\mathrm{qftp}_{\mathcal{L}_p}(z/U)$ is determined by $\mathrm{qftp}_{\mathcal{L}_S}(z/L)$. So, extend ρ by choosing a realization of $\mathrm{qftp}_{\mathcal{L}_S}(z/L)^\sigma$. !!!
- Case 2: $\alpha = x \in D \setminus U_D$... the "cover" part. The crucial part is that there is a finite set of elements of $S(F)$ whose coordinates generate L over the coordinates of $q(U_D)$ together with $E^{ab}(\Sigma)$. *Replace* $E^{ab}(\Sigma)$ with the extension generated by the coordinates of those elements and henceforth assume that U_S is generated by $q(U_D)$.
  - Subcase A: x is special - then only one choice for $\varphi(x)$.
  - Subcase B: The other case. Since S has dimension 1, may assume x is Hodge-generic. We knew (Prop. 3.1 of [1]) that $\rho\left(\mathrm{qftp}_{\mathcal{L}_p}(x/U)\right)$ is determined by again, clarify the context

$$\bigcup_{\overline{g}} \mathrm{qftp}_{\mathcal{L}_S}\left((q(g_1 x), \ldots, q(g_n x))/L\right)^\sigma.$$

A punchline: instead of considering directly $\mathfrak{H}/\Gamma_N$, we consider $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ and then $\mathfrak{H}^*/\Gamma$. This again is a Riemann surface, this time compact.

Therefore there exists $X_\Gamma \subset \mathbb{P}^m(\mathbb{C})$, a projective variety such that

$$\mathfrak{H}^*/\Gamma \approx X_\Gamma.$$

There is also an embedding

$$\mathfrak{H}/\Gamma \quad \hookrightarrow \quad \mathfrak{H}^*/\Gamma$$

and furthermore there exists Y_Γ an affine variety such that

$$Y_\Gamma \hookrightarrow X_\Gamma.$$

We simplify notation by denoting Y_{Γ_N} by $Y(N)$ and X_{Γ_N} by $X(N)$. The advantage of working with Y_N is that we have a good representation: its field of meromorphic functions has dimension 1 over $\mathbb{C}$; it is therefore (biholomorphic to) a curve. over which field?

Remark. Notice that

$$\mathrm{Aut}(Z_N/Z_1) = \mathrm{Aut}\begin{pmatrix} \mathfrak{H}/\Gamma_N \\ \downarrow \\ \mathfrak{H}/\Gamma_1 \end{pmatrix} = \Gamma_1/\Gamma_N = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Also,

$$\hat{\mathbb{C}} = \varprojlim_N \mathfrak{H}/\Gamma_N = \lim_{\overline{g}} \mathfrak{H}/\Gamma_{\overline{g}},$$

where $\overline{g} = (g_1, \dots, g_n) \in G^n$. Recall also that

$$\Gamma_N \hookrightarrow \Gamma_{\overline{g}} = g_1 \Gamma g_1^{-1} \cap \dots \cap g_n \Gamma g_n^{-1}.$$

The representation is then given by the "fundamental group", and by using Mumford-Tate.

$$\pi_1^1 = \mathrm{Aut}(\hat{\mathbb{C}}) \approx \varprojlim_N \Gamma_N/\Gamma_1 \approx SL_2(\hat{\mathbb{Z}}).$$

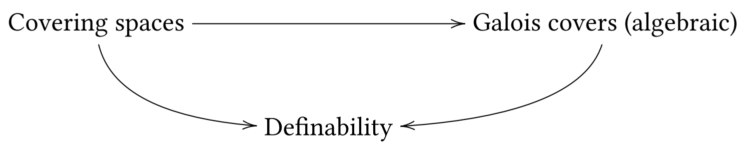
Remark. If an affine curve contains an infinite number of points defined over K then it is actually defined over K.

Theorem 4. *$Z_{\overline{g}}$ is a quasiaffine irreducible variety, defined over K; furthermore, the automorphism group Aut $(Z_{\overline{g}}/Z_1)$ is defined over K.*

Note also that $\mathrm{Gal}\left(\overline{\mathbb{Q}}^{ab}/\mathbb{Q}\right) = \hat{\mathbb{Z}}^* = \mathrm{Aut}\,(\text{Roots of Unity}) = \mathrm{Aut}\,(\mathrm{Tor}(S^1))$.

4. Around Definability (Leonardo Cano)

Recall, around categoricity of the j-function, the diagram of connections:

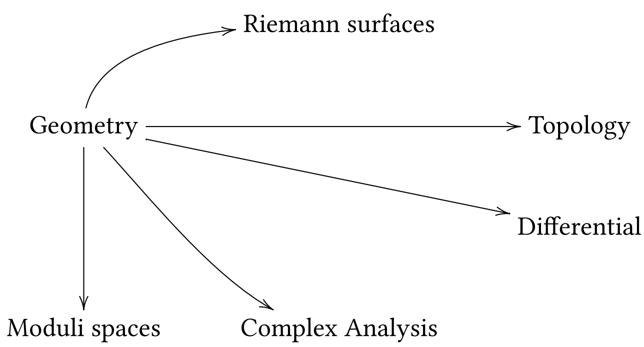


The aim of this section is to explain how to associate definability to covering spaces.

Remark. The main cover

$$\begin{array}{c} \mathfrak{H}/\Gamma_N \\ \downarrow \\ \mathfrak{H}/\Gamma_1 \end{array}$$

explored in the previous section brings connections between the following areas:



But how to turn covers into algebraic objects? The modular theorem seems to be needed in order to go to the algebraic part!

Let $\psi(N)$ be the number of cosets of Γ/Γ_N. Recall the mapping p_N:

$$\begin{array}{ccc} \Gamma_N & \xrightarrow{p_N} & \mathbb{C}^{\psi(N)+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ \mathfrak{H}/\Gamma & \xrightarrow[j]{} & \mathbb{C} \end{array}$$

given by $p_N(\tau) = \big(j(\tau), j(g_1\tau), \ldots, j(g_{\psi(N)}\tau)\big)$.

Our goal is to show that $p_N(\mathfrak{H})$ (or is it $p_N(\mathfrak{H}/\Gamma_N)$?) is definable in $(\mathbb{C}, +, \cdot, j(s_0))$. s_0 in $j^{-1}(\{0, 1728\})$. The point is that $p_N(\mathfrak{H})$ is the set of all Γ/Γ_N-torsors that contains $\big(j(s_0), \ldots, j(g_{\psi(N)}s_0)\big)$ ... + the topological theory of covering spaces ... give the proof.

Explain the role of s_0 in this...

The group $H < S_M$ acts by $\mathbb{C}^{\psi(N)+1}$ by permutation of indices such that $\gamma(x_0, \ldots, x_N) = (x_0, x_{\gamma(1)}, \ldots, x_{\gamma(N)})$.

$B_H := \big\{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid (x_1, \ldots, x_N) \text{ is an H-torsor}\big\}/H$.

Therefore Γ acts on $\mathfrak{H}/\Gamma_N$:

$$\begin{array}{ccc} \mathfrak{H}/\Gamma_N & \longrightarrow & \mathbb{C}^{\psi(N)+1} \\ \downarrow & & \downarrow \\ \mathfrak{H}/\Gamma & \longrightarrow & \mathbb{C} \end{array}$$

and

$$\begin{array}{ccc} \Sigma & \longrightarrow & \tilde{\Sigma}_H \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \times B_H \end{array}$$

Here we have $\tau \mapsto (\tau, \varphi(\tau)) \in \mathbb{C} \times B(H)$.

Also, a form of uniqueness: if

$$\begin{array}{ccc} \mathfrak{H}/\Gamma_N & \xrightarrow{\alpha_i} & \tilde{\Sigma}_H \\ \downarrow & & \downarrow \\ \mathbb{C} \approx \mathfrak{H}/\Gamma & \longrightarrow & \mathbb{C} \times B_H \end{array}$$

for $i = 1, 2$, then if $\alpha_1(s_0) = \alpha_2(s_0)$ we have that $\alpha_1 = \alpha_2$. It must then be possible to endow B_H with a global chart into $\mathbb{C}^{M+1}$ such that

$$\tau \mapsto (\varphi_1(\tau), \ldots, \varphi_{M+1}(\tau))$$

is built from *symmetric polynomial functions.* Then the "modular polynomial"

$$\phi_N(z, \tau) = \prod_{i=1}^{\psi(N)} (z - j(g_i\tau))$$

describes $p_N(\mathfrak{H})/\Gamma_N$.

Remark. Remember the g_i's come from the representation of $\Gamma N \Gamma$ as a union of cosets with generators $g_1, \ldots, g_n$. Fixing τ, the fiber is $j(g_1\tau), \ldots, j(g_{\psi(N)}\tau)$. This suggests that the space of torsors behaves as the "space of roots" of the modular polynomial $\phi_N(X, j) \in \mathbb{Z}[X, j]$ and

$$p_N(\mathfrak{H}) = \{z \mid \phi_N(z, \tau) = 0 \text{ for } \tau \in \mathbb{C}\}$$

5. Necessary Conditions and Keisler's Theorem

If S is a Shimura variety and $\mathbf{p}$ denotes the two-sorted structure associated (particular case, $\mathbf{j}$. If we fix $x_1, \dots, x_m \in X^+$ a collection of Hodge-generic points in different $G^{ad}(\mathbb{Q})^+$-orbits and consider $L = E^{ab}(\Sigma)(p(x_1), \dots, p(x_M))$.

The main goal of that section in [1] is to prove

Theorem 5. *If $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$ is categorical, then the image of the homomorphism*

$$Aut(\mathbb{C}/L) \to \overline{\Gamma}^m$$

(associated with $\overline{z} = (p(x_1), \dots, p(x_M))$) has finite index.

5.1. Keisler's Theorem.

One of few applications of infinitary logic to geometry, Keisler's theorem provides a bound on the number of complete types realizable for $\mathcal{L}_{\omega_1,\omega}$-sentences, under $\aleph_1$-categoricity.

Theorem 6. ***(Keisler)** If an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ is $\aleph_1$-categorical then the set of complete m-types realizable in models of ψ is at most countable.*

The proof of this uses some interesting argument in the model theory of infinitary logic. It has impact in Abstract Elementary Classes.

The idea of the proof of Theorem 5 is to try to build models of $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$ realizing uncountably many complete types.

The groups to use are our

$$\Gamma_{\overline{g}} = g_1^{-1}\Gamma g_1 \cap \dots \cap g_n^{-1}\Gamma g_n$$

for all tuples $\overline{g} = (g_1, \dots, g_n)$ of distinct elements of $G^{ad}(\mathbb{Q})^+$. Remember the inverse system of quotients $X^+/\Gamma_{\overline{g}}$ (locally symmetric varieties). This **system** carries an action of $G^{ad}(\mathbb{Q})^+$: the action of each $\alpha \in G^{ad}(\mathbb{Q})^+$ on X^+ induces a map explain loc. symm. var.

$$\Gamma_{\overline{g}} \setminus X^+ \to \alpha\Gamma_{\overline{g}}\alpha^{-1} \setminus X^+$$

(each variety is sent to the quotient by a "conjugate" - another variety)... Let $\underline{S}$ be its inverse limit - an equivalence class in $\Gamma_{\overline{g}} \setminus X^+$ is denoted by $[\cdot]_{\Gamma_{\overline{g}}}$. A point of $\underline{S}$ is a compatible collection of points $[x_{\overline{g}}]_{\Gamma_{\overline{g}}} \in \Gamma_{\overline{g}} \setminus X^+$ - the action of $G^{ad}(\mathbb{Q})^+$ on components is given by

$$[x_{\overline{g}}]_{\Gamma_{\overline{g}}} \mapsto [\alpha x_{\overline{g}}]_{\alpha\Gamma_{\overline{g}}\alpha^{-1}}.$$

Now let $\overline{x} \in \underline{S}$ with components $[x_{\overline{g}}]_{\Gamma_{\overline{g}}}$ which are images of Hodge-generic points $x_{\overline{g}} \in X^+$. The aim is to get a model

$$\mathbf{q} = \langle \mathbf{D}, \mathbf{S}, q \rangle$$

of $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$ and an $x \in D$ such that for all tuples $\overline{g} = (e, g_1, \dots, g_n)$, we have that $(q(x), q(g_1 x), \dots, q(g_n x)) \in Z_{\overline{g}}$ is equal to the image of $[x_{\overline{g}}]_{\Gamma_{\overline{g}}}$ in $Z_{\overline{g}}$ under the isomorphism

$$\Gamma_{\overline{g}} \setminus X^+ \to Z_{\overline{g}} \qquad [x_{\overline{g}}]_{\Gamma_{\overline{g}}} \mapsto (q(x_{\overline{g}}), q(g_1 x_{\overline{g}}), \dots, q(g_n x_{\overline{g}})).$$

Lemma 7. *The group $\Gamma_\infty := \cap_g \Gamma_g$ belongs to $Z_G(\mathbb{Q})$.*

Not clear if p or q there... This seems to mean that Mumford-Tate would hold...

Therefore, we can embed X^+ into $\underline{S}$ via $x \mapsto \left([x]_{\Gamma_g}\right)_g$.

The definition of $\mathbf{q}$[1]: the $\mathbf{D}$ sort of $\mathbf{q}$ is then the set D together with its action of $G^{\mathrm{ad}}(\mathbb{Q})^+$; the sort $\mathbf{S}$ is the algebraic variety $S(\mathbb{C})$ with relations for all Zariski-closed subsets of its cartesian powers defined over $E^{\mathrm{ab}}(\Sigma)$ and the map q is just the restriction of $\underline{p}$ to D.

Where do we use the lemma?

Lemma 8. *If $g\tilde{x} = \tilde{x}$ for some $g \in G^{ad}(\mathbb{Q})^+$, then g is the identity.*

By the completeness of $\mathrm{Th}(\mathbf{p})$, and by the QE, the previous lemma implies that $\mathbf{q} \in \mathrm{Th}(\mathbf{p})$.

Lemma 9. *The structure* $\mathbf{q}$ *also satisfies Standard Fibres.*

(Use Hodge-genericity.)

Now pick $x_1, \dots, x_m \in X^+$ a collection of Hodge-generic points in distinct $G^{\mathrm{ad}}(\mathbb{Q})^+$-orbits and let L be the field obtained by adjoining to $E^{\mathrm{ab}}(\Sigma)$ the coordinates of $\bar{z} := (p(x_1), \dots, p(x_m))$.

The following proposition seems to be crucial.

Proposition 10. *If the homomorphism*

$$\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma}^m$$

associated with $\bar{z}$ is of infinite index, then the set of complete m-types realizable in models of $Th^{\infty}_{SF}(\mathbf{p})$ is of cardinality at least $2^{\aleph_0}$.

Proof. We look at types over L. And we show that the number of types is bounded below by the index of Aut $(\mathbb{C}/L)$ in $\overline{\Gamma}^m$ and we do a binary tree construction.

Let

$$\mathbf{q} = \langle \mathbf{D}, S(\mathbb{C}), \mathcal{R}, q \rangle$$

be a model of $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$. Now consider an m-tuple $x'_1, \dots, x'_m \in D$ in the fiber given by $p(x_i)$ for $i = 1, \dots, m$. We want to analyze

$$\mathrm{tp}_{L_S}(x'_1, \dots, x'_m/L).$$

This is determined by

$$\bigcup_{\overline{g}} \mathrm{qftp}_{L_S}\,(q_{\overline{g}}(x'_1), \dots, q_{\overline{g}}(x'_m)/L)$$

($\overline{g} = (e, g_1, \dots, g_n)$ a tuple of distinct elements of $G^{\mathrm{ad}}(\mathbb{Q})^+$.

In case $m = 1$, the projection $S(\mathbb{C})^{n+1} \to S(\mathbb{C})$ restricted to $Z_{\overline{g}}$ is a finite morphism corresponding to the natural map

$$\Gamma_{\overline{g}} \setminus X^+ \to \Gamma \setminus X^+.$$

Now, the qf type $\mathrm{qftp}_{L_S}\,(q_{\overline{g}}(x'_1)/L)$ is determined by the minimal algebraic subset of $Z_{\overline{g}}$ containing $q_{\overline{g}}(x'_1)$... this is a subset of the fiber over $q(x'_1)$ of the finite morphism above... so, 0-dimensional. It is indeed the Aut $(\mathbb{C}/L)$-orbit of this fiber containing $q(x'_1)$. Similarly for arbitrary m (check).

Now, the number of orbits is equal to the index of the image of Aut $(\mathbb{C}/L)$ in $\Gamma^m/\Gamma^m_{\overline{g}}$. Look at the ordering of tuples of the form $\overline{g} := (e, g_1, \dots, g_n)$ by extension. As you "move

[1]We use for this $\underline{p} : \underline{S} \to S(\mathbb{C})$ the natural map, and let D be the union of $\mathcal{O} := \left\{ g\tilde{x} \mid g \in G^{\mathrm{ad}}(\mathbb{Q})^+ \right\}$ and $X^+ \setminus \left\{ x \in X^+ \mid \underline{p}(x) \in \underline{p}(\mathcal{O}) \right\}$.

up" this order, you take a successive tuple for which we obtain another index - equal to a multiple of the previous.

Either the number stabilizes or it continues to increase in at least multiples of two.

In particular, the index of

$$\mathrm{Aut}\,(\mathbb{C}/L) \to \overline{\Gamma}^m$$

is either finite or $2^{\aleph_0}$.

Every possible type is indeed realized in some model - by the previous construction of models of $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$. □

This, combined with Keisler's theorem, gives the main theorem, Theorem 5.

Proof. (Of Theorem 5) If $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$ is $\aleph_1$-categorical, by Proposition 10 if

$$\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma}^m$$

is not of finite index, then the set of complete m-types realized in models of $\mathrm{Th}^{\infty}_{SF}(\mathbf{p})$ is uncountable. Keisler's theorem (Theorem 6 here) says that this is impossible, under $\aleph_1$-categoricity. Then $\left(\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma}^m\right)$ is of finite index. □

This suggests...

What about beyond $\mathcal{L}_{\omega_1,\omega}$?

6. Dos Preguntas sobre las ideas de Harris y Moonshine - ¿nuevas funciones tipo j?

Lo siguiente está basado en comunicación de Jorge Plazas sobre categoricidad.

6.1. **Sobre categoricidad.** Sea $\Gamma \leqslant GL_2(\mathbb{Q})$ tal que para algún entero N se tiene que $\Gamma(N)$ es un subgrupo de índice finito de Γ (así, $\Gamma(N)$ debe ser un subgrupo discreto de Γ, conmensurable con $SL_2(\mathbb{Z})$. En este caso los puntos parabólicos (cusps) de Γ coinciden con los de $SL_2(\mathbb{Z})$ y $X_\Gamma := \Gamma \setminus \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ tiene estructura de superficie de Riemann compacta.

- Si X_Γ es de género 0 entonces existe una función f_Γ (¡única módulo la adición de una constante!) que nos da isomorfismo de X_Γ con la esfera de Riemann $\mathbb{P}^1(\mathbb{C})$. Con esta notación,
$$j(\tau) = f_{SL_2(\mathbb{Z})}(\tau).$$
Ahora, dado Γ como este, con género cero, **¿es categórica la teoría de f_Γ?**
- Dado Γ uno de los 163 grupos de género cero que dan caracteres graduados del monstruo, **¿es categórica la teoría de f_Γ? ¿Se pueden combinar todas las 163 funciones en una sola teoría?**

Jorge cree que la pregunta 1 ya está resuelta en Daw-Harris [4].

6.2. **Teoría de modelos del monstruo y sus representaciones.** En caso de respuesta positiva a la pregunta 2 arriba, **¿se puede construir una teoría a partir del monstruo y sus representaciones?** Esto de manera que... ¡sin apelar a Moonshine se pueda llegar a funciones que sean modelos de la (hipotética por ahora) teoría de las 163 funciones f_Γ de moonshine!

Si la teoría (hipotética) resulta categórica, estas funciones serían isomorfas y (en ese caso) *se tendría una demostración modelo-teórica de moonshine.*

Jorge: ¿entre sí? ¿a alguna más "canónica"?

7. Temas relacionados pero por ahora aislados

Otros temas discutidos durante el seminario - y mencionados en la carta de Jorge Plazas, son los siguientes (por ahora mucho más aislados que lo anterior).

7.1. **Cohomología étale de Grothendieck.** Leonardo ha indagado más a fondo sobre la conexión entre espacios recubridores y formas de categoricidad - mediadas por la representación. En realidad (dice Jorge) esto hace parte de temas más generales de cohomología étale de Grothendieck (algo que he estado mirando también en conexión con mi trabajo con Padilla sobre cohomología modelo-teórica y Cruz sobre esquemas de Zariski) - Jorge sugiere revisar Milne [2] para bases sobre el tema.

7.2. **Función-j cuántica y multiplicación real.** En este tema parece haber diversidad de miradas:

- Gendron y sus colaboradores tienen variantes reales de la función-j dadas por la llamada "j universal" (en realidad una sección en un haz sobre un espacio de Stone asociado a "pendientes reales" en modelos no estándar - una manera de capturar la aproximación diofantina de un real y asociarla de manera natural a la función-j clásica. Algebraicidad sigue siendo un problema abierto, pero recientemente con Luca Demangos afirman haber demostrado la algebraicidad de valores de una variante p-ádica de la función-j, cuando se calcula en una irracionalidad cuadrática.
- Jorge sugiere hacer algo un poco distinto: mirar lo que se ha hecho hasta ahora sobre generadores explícitos de cuerpos de clases de cuerpos cuadráticos reales. El primer "test" que debe pasar cualquier función-j sobre la recta real es tener valores algebraicos en irracionalidades cuadráticas. Manin y Marcolli han sugerido revisar símbolos modulares que son clases de homología en curvas modulares dadas por geodésicas sobre estas - y extenderlas a la frontera real. (Agrega AV: hasta ahora parece también poco conclusivo esto).

8. Notas para sesión lunes 14 de marzo

8.1. **Cubierta pro-étale.**

- Recordar
$$\hat{\mathbb{C}} := \varprojlim_{g \subset G} Z_g.$$

Esto es pro-definible en $\langle \mathbb{C}, +, \cdot, \mathbb{Q}(j(S)) \rangle$. Esto viene con
$$\hat{j} : \hat{\mathbb{C}} \to \mathbb{C}.$$

Cubierta universal con respecto al subsistema de cubrimientos correspondiente a subconjuntos finitos $g \subset G$. $\hat{\mathbb{C}}$ viene equipado con un punto base para el levantamiento $(p_g(s_0))_g$. Las notas sobre definibilidad usan esto.

- Recordar la acción de Galois sobre $\hat{\mathbb{C}}$. Dada $q_N : Z_N \to Z_1$, esto induce una acción a izquierda de
$$\mathrm{Aut}_{\mathrm{Fin}}(Z_N/Z_1) \approx \Gamma/\Gamma_N$$
sobre la fibra $q_N^{-1}(x)$. En el límite $\pi_1' := \varprojlim_N \mathrm{Aut}_{\mathrm{Fin}}(Z_N/Z_1)$ la fibra $\hat{j}^{-1}(x)$ es un π_1'-torsor. Si M divide a N entonces el mapa entre $\mathrm{Aut}_{\mathrm{Fin}}(Z_N, Z_1)$ y $\mathrm{Aut}_{\mathrm{Fin}}(Z_M, Z_1)$ depende de s_0.
- Recordar la acción a izquierda de $\mathrm{Aut}(\mathbb{C}, +, \cdot, 0, 1) \approx \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ dada por $(x_1, \cdots, x_n)^\sigma := (x_1^\sigma, \cdots, x_n^\sigma)$.

9. Notes for a general seminar lecture (BCC-CUNY)

For the seminar at Bronx Community College: a general (mathematical) audience, with number theorists and logicians.

- Option 1: start by the *statement* of a theorem and explain the logic (definability, types, etc.) behind.
- Option 2: follow some older lecture... no!
- Option 3: base the lecture on questions driven by the j-function...

Of all these, perhaps the best is option 3. Option 1 worked at Los Andes but here there will be more people knowledgeable in number theory (there only Mantilla...). The questions and partial solution idea seems good.

- Define j - give the series version
-

References

[1] Adam Harris. Categoricity of the two sorted j-function, 2013.
[2] John Milne. Lectures on etale cohomology, 2008.
[3] Richard Taylor. Galois representations, 2003.
[4] Christopher Daw y Adam Harris. Categoricity of modular and shimura curves, 2015.