Dos lógicas extrañas, grandes cardinales y algo de forcing

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Seminario de Topología y Teoría de Conjuntos / UNAM (Morelia), 2022

Contents

When is a logic "appropriate" for model theory?

(A natural question, perhaps)…some answers.

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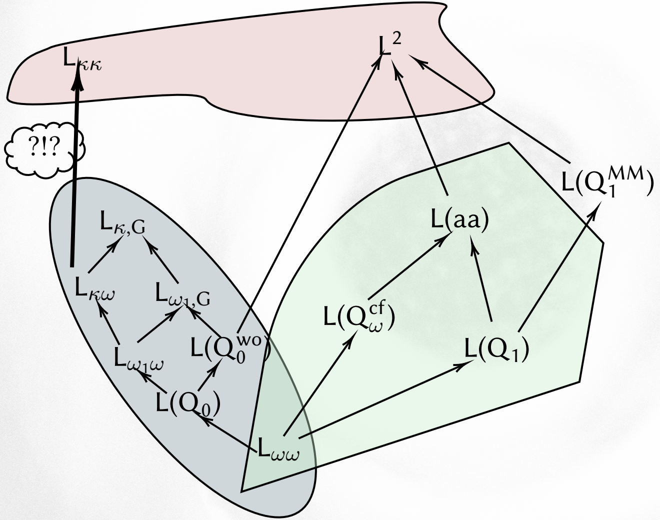
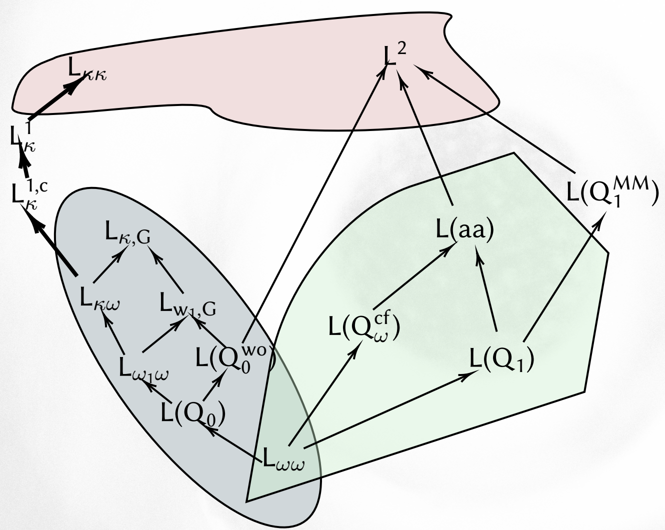
- $\mathsf{L}_{\omega_1,\omega}$? Compactness fails.
- $\mathsf{L}_{\kappa,\lambda}$…It depends strongly on κ (and λ)
- Väänänen says: "infinitary logic may still serve as a 'yardstick' for model theoretic constructs, permits fragments of model theory and is preserved under (reasonable) forcing"…

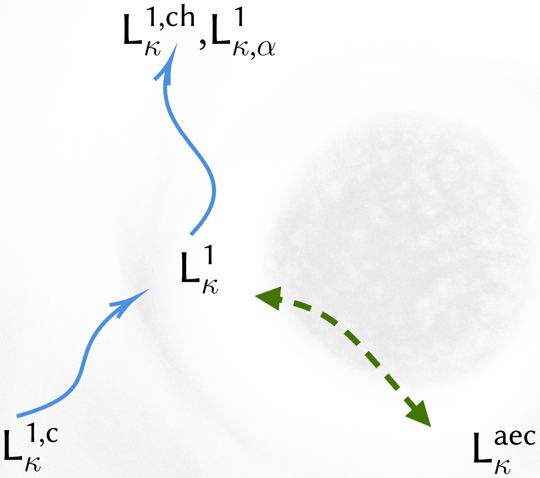
A map of various infinitary logics



New Logics



L^2
$L_{\kappa\kappa}$
L^1_κ
$L^{1,c}_\kappa$
$L_{\kappa,G}$
$L_{\kappa\omega}$
$L_{w_1,G}$
$L_{\omega_1\omega}$
$L(Q_0^{wo})$
$L(Q_0)$
$L_{\omega\omega}$
$L(Q_\omega^{cf})$
$L(aa)$
$L(Q_1)$
$L(Q_1^{MM})$


$\mathbb{L}_{\kappa}^{1,ch}$, $\mathbb{L}_{\kappa,\alpha}^{1}$
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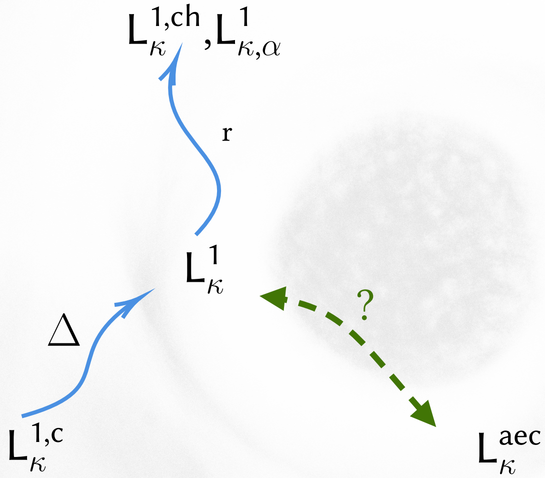
$\mathbb{L}_{\kappa}^{aec}$


$L^{1,ch}_\kappa$, $L^1_{\kappa,\alpha}$
r
L^1_κ
Δ
?
$L^{1,c}_\kappa$
L^{aec}_κ

Interpolation issues

- Craig$(\mathbb{L}_{\kappa^+\omega}, \mathbb{L}_{(2^\kappa)^+\kappa^+})$ (Malitz 1971).

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  If $\varphi \vdash \psi$, where φ is a τ_1-sentence and ψ is a τ_2-sentence and both are in $L_{\kappa^+\omega}$ then
  there exists $\chi \in L_{(2^\kappa)^+\kappa^+}(\tau_1 \cap \tau_2)$ such that

$$\varphi \vdash \chi \vdash \psi.$$

- The original argument used "consistency properties". Other proofs have stressed the "Topological Separation" aspect of Interpolation.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

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- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

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  and Craig(L^1_κ).

- Moreover, in the case $\kappa = \beth_\kappa$, the logic L^1_κ also has a Lindström-type characterization as the maximal logic with a peculiar strong form of undefinability of well-order.

A first description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.

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- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?
- There are at least three partial answers, one approaching from below (Väänänen-V.), the other one from above (Džamonja, Väänänen), a third one modifying the length and the clock of the game (Velickovic, Väänänen). We will focus on the first one.

Shelah's game $\eth^{\beta}_{\theta}(M, N)$.

ANTI	ISO
$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega, g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^1} \to \omega, g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

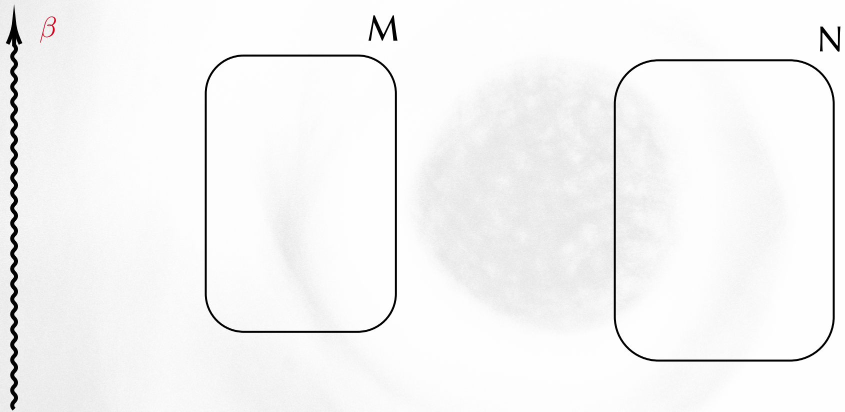
Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2n}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

ISO wins if she can play all her moves, otherwise ANTI wins.

- $\mathsf{M} \sim_\theta^\beta \mathsf{N}$ iff ISO has a winning strategy in the game.
- $\mathsf{M} \equiv_\theta^\beta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim_\theta^\beta \mathsf{N}$.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L_κ^1.

Shelah's game $\Game^{\beta}_{\theta}(M, N)$.



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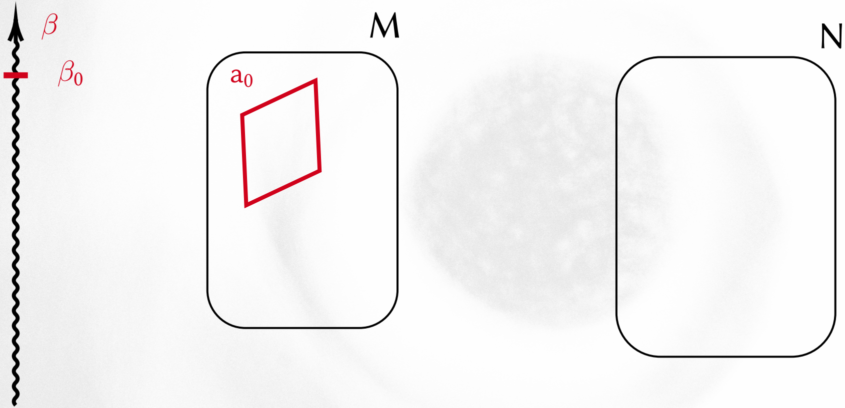

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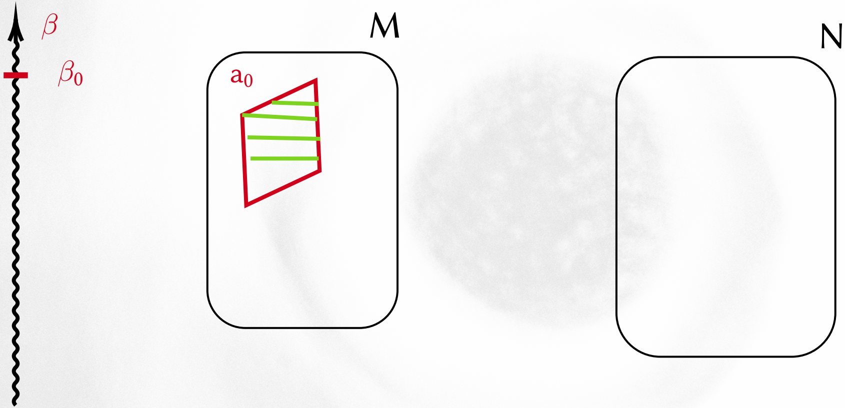

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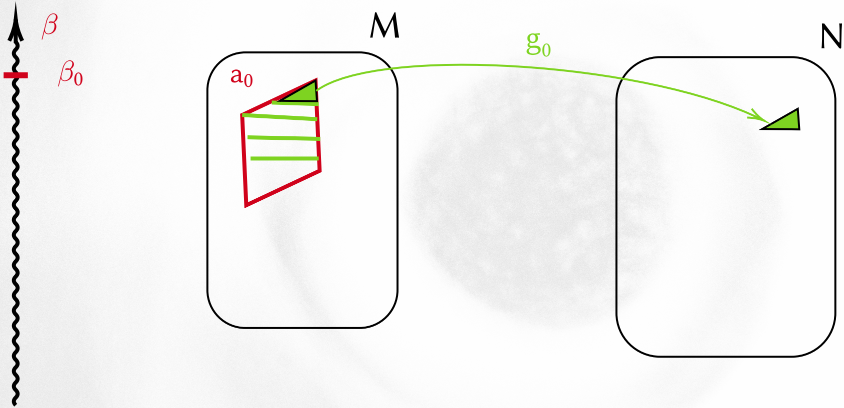

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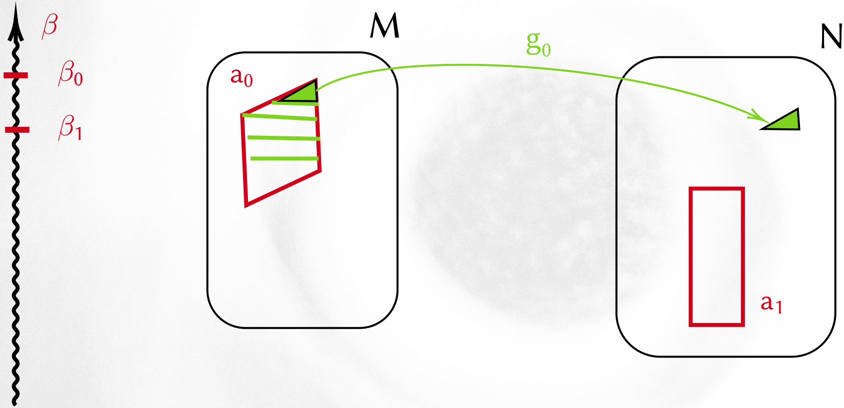

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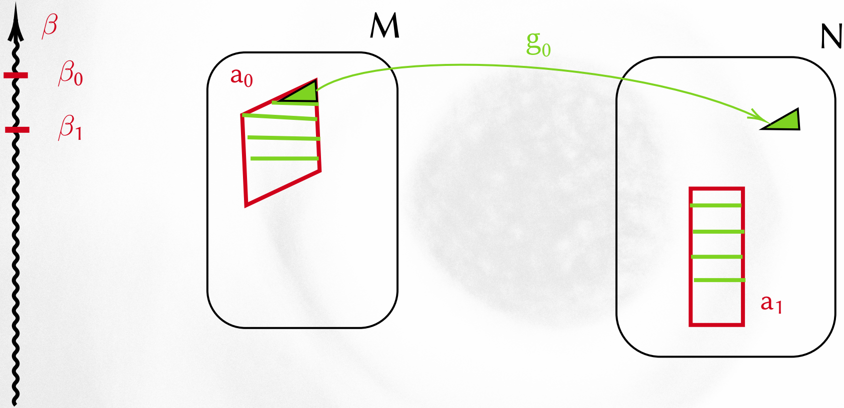
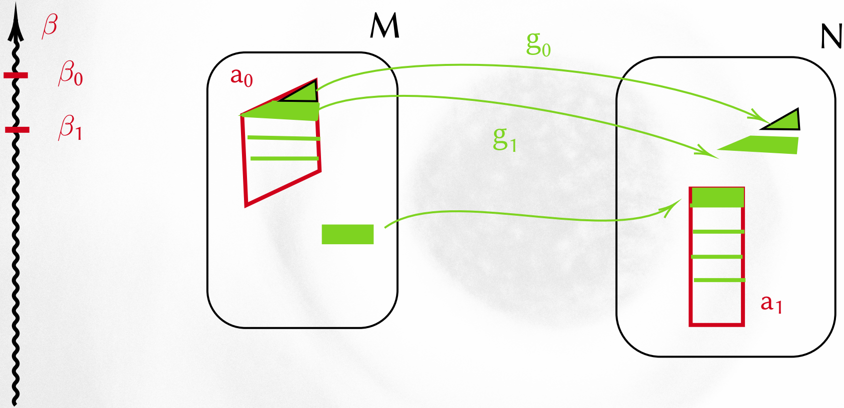


Shelah's game $\partial^{\beta}_{\theta}(M, N)$.



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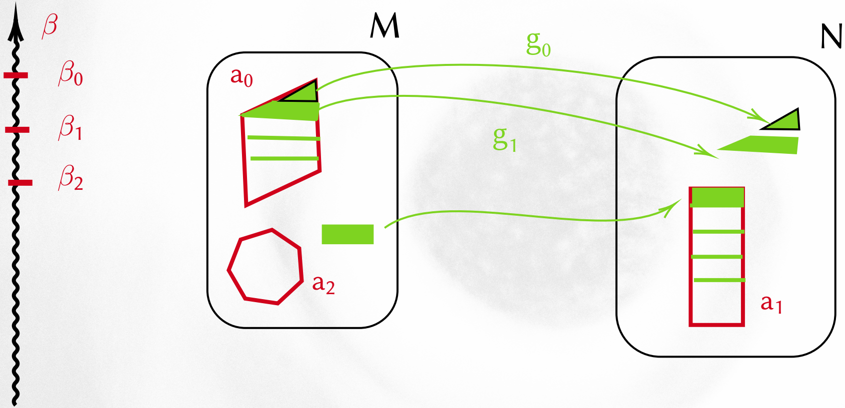

The definition of L^1_κ-sentences - again

- For M, N τ-structures, θ a cardinal, $\alpha \leq \theta$ an ordinal, $\mathsf{M} \sim^\beta_\theta \mathsf{N}$ iff ISO has a winning strategy in $\eth^\beta_\theta(\mathsf{M}, \mathsf{N})$,
- $\mathsf{M} \equiv^\beta_\theta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim^\beta_\theta \mathsf{N}$,
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv^\beta_\theta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L^1_κ.

Comparison with other logics: where is L^1_κ?

$$\bigcup_{\lambda<\theta} \mathsf{L}_{\lambda^+,\omega} \leq \mathsf{L}^1_{\leq\theta} \leq \bigcup_{\lambda<\beth_{\theta^+}} \mathsf{L}_{\lambda^+,\lambda^+}$$

Key Lemma for second dominance:

$$\mathsf{M}_1 \equiv_{\mathsf{L}_{\beth_\beta(\theta)^+,\theta^+}} \mathsf{M}_2 \ (\forall\beta<\theta) \implies \mathsf{M}_1 \sim_{\eth^{<\theta^+}_{\leq\theta}} \mathsf{M}_2$$

(Induction on β: if **s** is a state in $\eth^{<\theta^+}_{\leq\theta}$, $\varphi(\bar{x})$ is a formula of $\mathsf{L}_{\beth_\beta(\theta)^+,\theta^+}$ such that

$$\mathsf{M}_1 \models \varphi[\mathrm{dom}(g_s)] \leftrightarrow \mathsf{M}_2 \models \varphi[\mathrm{ran}(g_s)]$$

then **s** is a winning state for ISO in $\eth^{<\theta^+}_{\leq\theta}$.)

Crucial claim: closure under unions of ω-chains

Given $(M_n)_{n<\omega}$ a sequence of τ-structures and given $\psi(\bar{z}) \in L^1_{\leq\theta}(\tau)$, if

$$M_n \prec_{L_{\partial^+,\theta^+}} M_{n+1}\text{, for all } n < \omega, \partial = \beth_{\theta^+}$$

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$$M_n \prec_{L_{\partial^+,\theta^+}} M_{n+1}\text{, for all } n < \omega, \partial = \beth_{\theta^+}$$

then

$$M_n \equiv_{L^1_\theta} M_\omega := \bigcup_{n<\omega} M_n$$

and

$$\forall \bar{a} \in {}^{\lg(z)}M_0 \quad M_n \models \psi[\bar{a}] \Leftrightarrow M_\omega \models \psi[\bar{a}] \text{ for all } n < \omega.$$

(Weak) Downward Löwenheim-Skolem for L^1_κ

Assuming $\kappa = \beth_\kappa$,
for every sentence $\psi \in \mathsf{L}^1_\kappa$, if there exists M such that $\mathsf{M} \models \psi$ then there exists a model $\mathsf{N} \models \psi$, N of cardinality $< \kappa$.
for every $\psi \in \mathsf{L}^1_\kappa$ there is $\partial < \kappa$ such that: if N is a model of ψ of cardinality λ and $\mu = \mu^{<\partial}$ then some submodel M of N of cardinality μ is a model of ψ

Undefinability of Well-Ordering

For this, the assumption $\kappa = \beth_\kappa$ seems crucial all along.
SUDWO (Strong Undefinability of Well Ordering):
If $\psi \in \mathsf{L}^1_\kappa(\tau)$, $|\tau| < \kappa$, $<$, R are binary predicates, $\mathsf{c}_1, \mathsf{c}_2$ constants from τ, THEN for every large enough $\mu_1 < \kappa$ for arbitrarily large $\mu_2 < \kappa$ we have:
if $\lambda > \mu_2$, $\mathfrak{A}$ is a τ-expansion of $(\mathsf{H}(\lambda), \in, \mu_1, \mu_2, <)$, with $<$ the order on ordinals, $\mathsf{R}^{\mathfrak{A}}$ being $\in$, $\mathsf{c}_1^{\mathfrak{A}} = \mu_1$, $\mathsf{c}_2^{\mathfrak{A}} = \mu_2 \ldots$ then there is $\mathfrak{B}$, a_n, d_n $(\mathsf{n} < \omega)$ such that

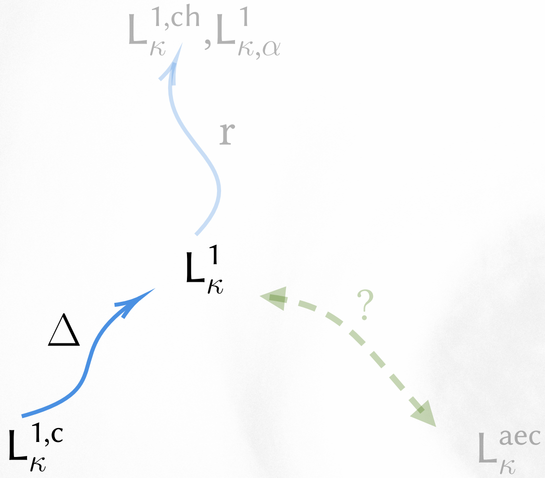
- $\mathfrak{B} \models \psi \Leftrightarrow \mathfrak{A} \models \psi$,
- $\mathfrak{B} \models \mathsf{d}_{\mathsf{n}+1} < \mathsf{d}_\mathsf{n} < \mu_2$ for $\mathsf{n} < \omega$,
- $\mathfrak{B} \models \mathsf{a}_\mathsf{n} \subseteq \mathsf{a}_{\mathsf{n}+1}$ has cardinality $\leq \mu_1$,
- if $\mathsf{e} \in \mathfrak{B}$ and $\mathfrak{B} \models |\mathsf{e}| \leq \mu_1$ then $\mathfrak{B} \models \mathsf{e} \subseteq \mathsf{a}_\mathsf{n}$ for some n

A Lindström-like theorem

Let $\mathcal{L}$ be "a logic", let $\kappa = \beth_\kappa$. If $\mathcal{L}$ satisfies the following properties:

- $\mathcal{L}$ is nice (natural closure properties),
- the occurrence number of $\mathcal{L}$ is $\leq \kappa$,
- $\mathsf{L}_{\theta^+,\omega} \leq \mathcal{L}$, for $\theta < \kappa$,
- $\mathcal{L}$ satisfies SUDWO,

THEN
$\mathcal{L} \leq \mathsf{L}^1_\kappa$.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?
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Approaching L^1_κ from below (mod Δ)

- Joint work with **J. Väänänen**
- We define a sublogic $\mathsf{L}^{1,c}_\kappa$ of L^1_κ ("Cartagena Logic"),
- $\mathsf{L}^{1,c}_\kappa$ has a recursive syntax.
- Many (but not all) of the nice properties of L^1_κ also hold for $\mathsf{L}^{1,c}_\kappa$,
- The "distance" between the two logics is not large (Δ).

Syntax of $\mathsf{L}^{1,c}_{\kappa}$

Suppose $2^\theta < \kappa$. The formulas of $\mathsf{L}^{1,c}_{\kappa,\theta}$ are built from atomic formulas and their negations by means of the operation $\bigwedge_I, \bigvee_I$, where $|I| < \kappa$, and the following two operations:

Syntax of $L^{1,c}_{\kappa}$

Suppose $2^{\theta} < \kappa$. The formulas of $L^{1,c}_{\kappa,\theta}$ are built from atomic formulas and their negations by means of the operation $\bigwedge_I, \bigvee_I$, where $|I| < \kappa$, and the following two operations:
Suppose $\phi_A(\vec{x}, \vec{y})$, $A \subseteq \theta$, are formulas of $L^{1,c}_{\kappa,\theta}$ such that of the variables $\vec{x} = \langle x_\alpha : \alpha < \theta \rangle$ only those x_α for which $\alpha \in A$ occur free in $\phi_A(\vec{x}, \vec{y})$.

$$\forall \vec{x} \bigvee_{f} \bigwedge_{n} \phi_{f^{-1}(n)}(\vec{x}, \vec{y})$$

$$\exists \vec{x} \bigwedge_{f} \bigvee_{n} \phi_{f^{-1}(n)}(\vec{x}, \vec{y}),$$

where $\vec{x} = \langle x_\alpha : \alpha < \theta' \rangle$, $\theta' \leq \theta$ and $f : \theta' \to \omega$.

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where $\vec{x} = \langle x_\alpha : \alpha < \theta' \rangle$, $\theta' \leq \theta$ and $f : \theta' \to \omega$.

$$L^{1,c}_{\kappa} = \bigcup_{\theta < \kappa} L^{1,c}_{\kappa,\theta}$$

Subformulas of such formulas are the $\phi_A(\vec{x}, \vec{y})$, where $A \subseteq \theta'$. Thus the number of subformulas of such a formula is $2^{|\theta'|}$.

Cardinality quantifiers may be captured: $|P| < \theta$

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_{f} \bigwedge_{n} \Big(\bigwedge_{f(i)=n} P(x_i) \to \bigvee_{i \neq j \in f^{-1}(n)} (x_i = x_j) \Big)$$

says $|P| < \theta$.

An example of expressive power: no long chains

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_{f} \bigwedge_{n} \bigwedge_{i \neq j \in f^{-1}(n)} \neg x_i < x_j$$

says $<$ has no chains of length θ.

A covering property: the combinatorial core of L^1_κ!

The combinatorial core of Shelah's L^1_κ is captured by $L^{1,c}_\kappa$...

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$ and $\mathrm{len}(\vec{y}) = \omega$. The sentence

$$\forall \vec{x} \bigvee_{f} \bigwedge_{n} \exists \vec{y} \bigwedge_{g} \bigvee_{m} \bigwedge_{f(i)=n} \bigvee_{g(j)=m} R(y_j, x_i)$$

says every set of size $\leq \theta$ can be covered by countably many sets of the form $R(a, \cdot)$.

Corollary

Suppose $\theta < \kappa$. There is a sentence in $L^{1,c}_\kappa$ which has a model of cardinality θ if and only if $\theta^\omega = \theta$.

The EF-game of $L^{1,c}_{\kappa}$: $\mathfrak{D}^{\beta,c}_{\theta}(M,N)$.

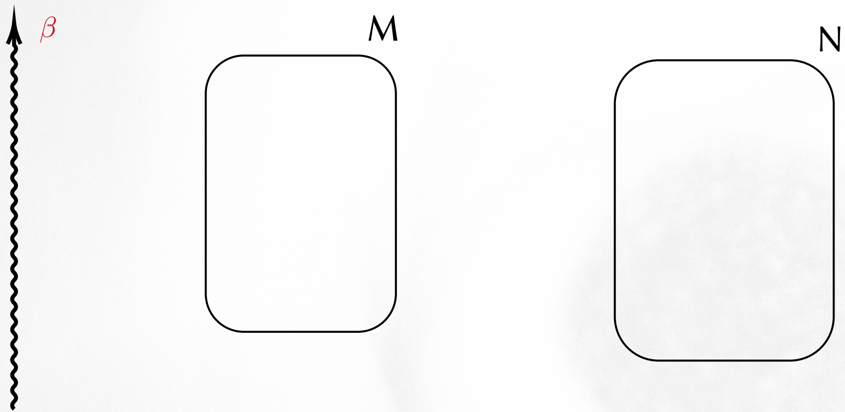
$\beta_0 < \beta$, $\vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$
$n_0 < \omega$	
	$g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0$, $\vec{a^1}$	
	$f_1 : \vec{a^1} \to \omega$,
$n_1 < \omega$	
	$g_1 : M \to N$ a p.i. $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\mathrm{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2i}(n_{2i}) \subseteq \mathrm{dom}(g_{2i})$
- $f^{-1}_{2i+1}(n_{2i+1}) \subseteq \mathrm{ran}(g_{2i})$.

Player II wins if she can play all her moves, otherwise Player I wins.

Our "Cartagena" game $\eth_\theta^{\beta,c}(M, N)$.



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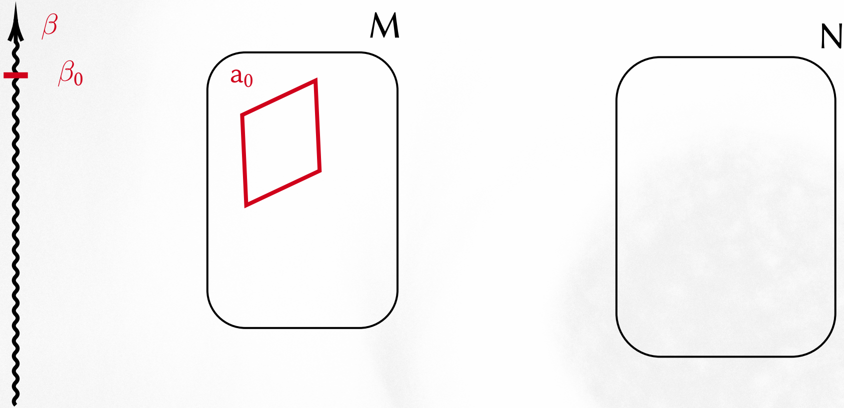

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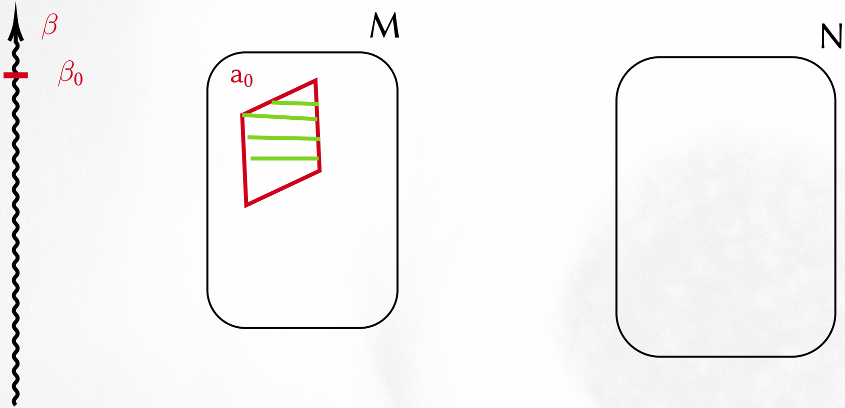

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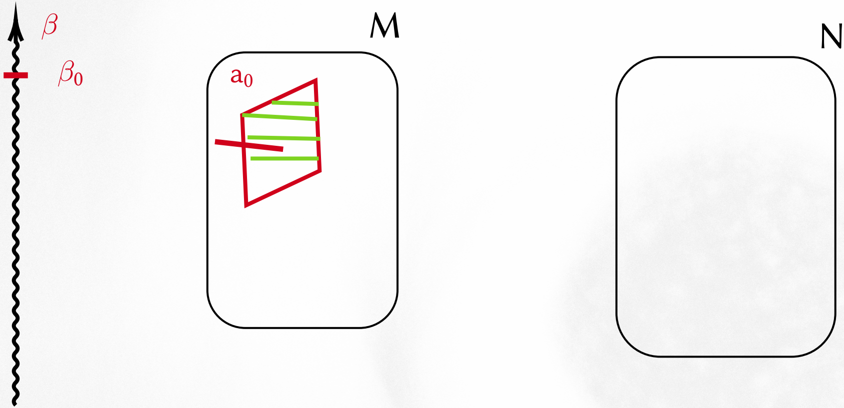

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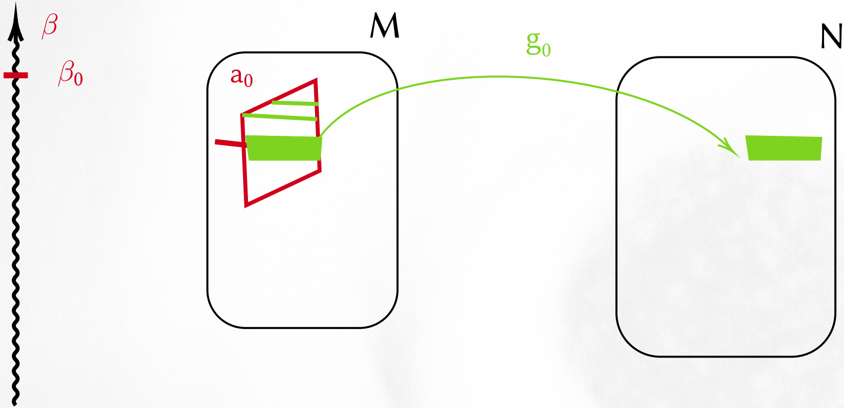

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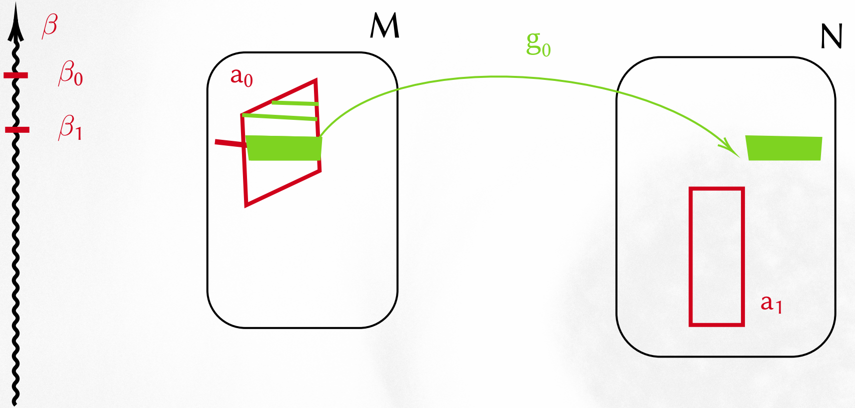

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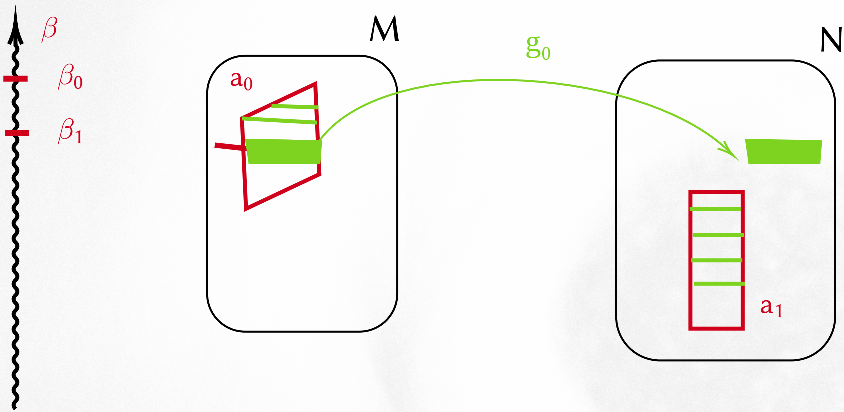

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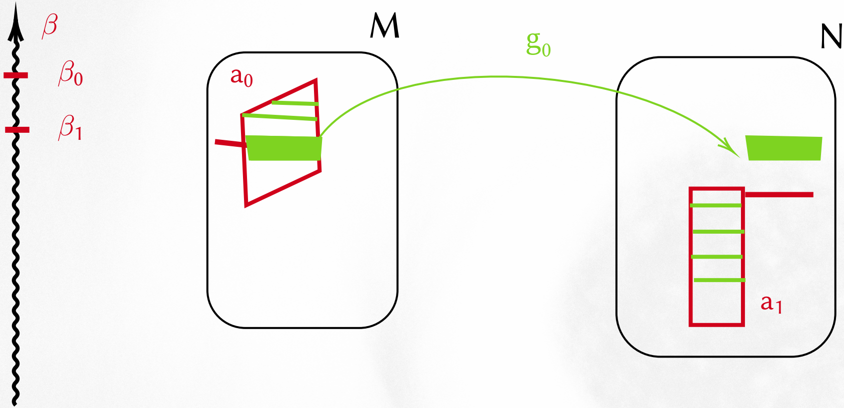

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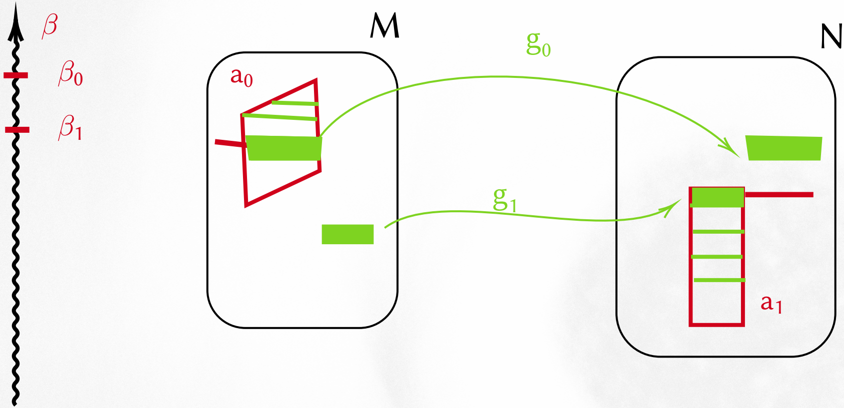

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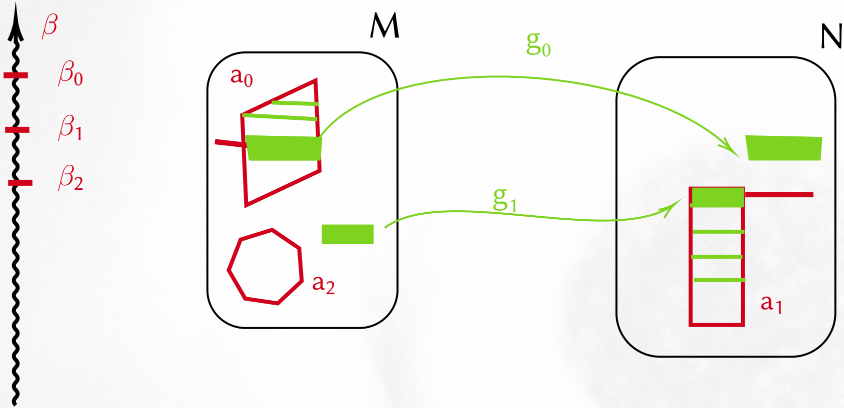

Theorem

The following are equivalent:

1. *Player II has a winning strategy in* $\partial_\theta^{\beta,c}(M, N)$.
2. M *and* N *satisfy the same sentences of* $L_{\theta^+}^{1,c}$ *of quantifier rank* $\leq \beta$.

Corollary

$L_\kappa^{1,c} \leq L_\kappa^1$.

Theorem

Assume $\kappa = \beth_\kappa$. *Then* $\Delta(L_\kappa^{1,c}) = L_\kappa^1$.

What is $\Delta(\mathsf{L})$?

- A model class $\mathcal{K}$ is $\Sigma(\mathsf{L})$ if it is the class of relativized reducts of an L-definable model class.
- A model class $\mathcal{K}$ is $\Delta(\mathsf{L})$ if both $\mathcal{K}$ and its complement are $\Sigma(\mathsf{L})$.
- $\Delta(\mathsf{L}_{\omega\omega}) = \mathsf{L}_{\omega\omega}$
- $\Delta(\mathsf{L}_{\omega_1\omega}) = \mathsf{L}_{\omega_1\omega}$
- $\Delta(\Delta(\mathsf{L})) = \Delta(\mathsf{L})$
- Δ preserves compactness, axiomatizability, Löwenheim-Skolem properties…

Union Property of $\mathsf{L}^{1,c}_{\kappa}$

Suppose Γ is a fragment of $\mathsf{L}^{1,c}_{\kappa}$, i.e. a set of formulas closed under subformulas.

$M_n \prec_\Gamma M_{n+1}$ means that for formulas $\varphi(\bar{x})$ in Γ and $\bar{a} \in M_n$ we have

$$M_n \models \varphi(\bar{a}) \qquad \rightarrow \qquad M_{n+1} \models \varphi(\bar{a}).$$

Lemma (Union Lemma)

If $M_n \prec_\Gamma M_{n+1}$ *for all* $n < \omega$, *then* $M_n \prec_\Gamma M_\omega$ *where* $M_\omega = \bigcup_n M_n$.

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"Hard direction:" $M_n \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$ implies $M_\omega \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$.
So let $A \in [M_\omega]^\theta$, $\theta < \kappa$. **We treat** $A \cup M_m$ **separately** for each m.
Since $M_m \models \forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f^{-1}(n)}(\bar{x}, \bar{a})$, there is $f_m : A \cap M_m \to \omega$ such that $M_m \models \bigwedge_n \varphi_{f_m^{-1}(n)}(A \cap M_m, \bar{a})$.

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Claim

$M_\omega \models \varphi_{f^{-1}(m)}(A \cap f^{-1}(m), \bar{a})$.

But this follows from the Induction Hypothesis as $A \cap f^{-1}(m) = A \cap f_k^{-1}(m')$ for some m', k and $M_k \models \varphi_{f_k^{-1}(m')}(A \cap f_k^{-1}(m'), \bar{a})$. □

A consequence of the Union Lemma

Theorem
Assume $\kappa = \beth_\kappa$. *Then* $\Delta(\mathsf{L}^{1,c}_\kappa) = \mathsf{L}^1_\kappa$.

Further properties include

- LS theorems
- Undefinability of well order
- $\Delta(\mathsf{L}^c_\kappa)$ contains any logic that satisfies the Union Lemma for $\prec_{\theta^+\theta^+}$, for arbitrary large $\theta < \kappa$. Shelah's L^1_κ is one such logic.

Note: Undefinability of well-order is a consequence of the LS property and the Union Lemma.

The advantages of $\mathsf{L}^{1,c}_\kappa$

- Simple syntax.
- Can express what L^1_κ does, at least implicitly.
- Its Δ-extension has Craig and Lindström Theorem.
- Undefinability of well-ordering is (also) a consequence of Caicedo's theorem on rigid structures and Uniform Reducibility of Pairs.

Plan

Virtually Large

- Schindler (2000): remarkable cardinals are equiconsistent with "$\mathsf{Th}(\mathsf{L}(\mathbb{R}))$ cannot be changed by proper forcing."

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can be "virtualized" by requiring the embedding to exist in a set-forcing extension of V.
- Virtual(ized) large cardinals are still large cardinals, but are now in the neighborhood of an ω-Erdős cardinal; they are consistent with L.

Virtually Large Cardinals

- A cardinal κ is **virtually supercompact** (remarkable) if for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive M with ${}^{\lambda}M \subseteq M$ such that there is a virtual elementary embedding $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

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Back to logic: the strong compactness cardinal of a logic

In 1971, Magidor proved that extendible cardinals are **strong compactness cardinals** for second-order infinitary logic $\mathsf{L}^2_{\kappa,\kappa}$. This means that every $< \kappa$-satisfiable theory **in this logic** is satisfiable.

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In their preprint Model Theoretic Characterizations of Large Cardinals, Boney, Dimopoulos, Gitman and Magidor [BDGM] generalize Magidor's early result to virtually extendible cardinals.

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*κ is virtually extendible iff every $<\kappa$-satisfiable $\mathsf{L}^2_{\kappa,\kappa}$-theory has a… **pseudo-model**.*

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Theorem (BDGM)

*κ is virtually extendible iff every $< \kappa$-satisfiable $\mathsf{L}^2_{\kappa,\kappa}$-theory has a... **pseudo-model**.*

They introduce the filtering of "being a model" (compactness) to "being a pseudo-model" (pseudo-compactness) and get the equivalence with virtuality.

Pseudo-models and forth-systems

So…what are these “filtered” models?

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Definition

Let T be a τ-theory in some logic $\mathcal{L}$, let M be a τ^*-structure.
A **forth system** $\mathcal{F}$ from τ to τ^* is a collection of renamings $\mathsf{f} : \sigma \to \sigma^*$, with σ, σ^* finite subsets of τ, τ^* respectively, such that

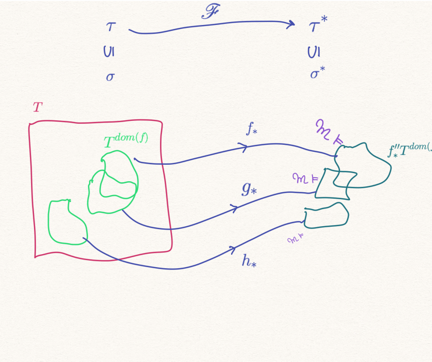
1. $\emptyset \in \mathcal{F}$,
2. If $\mathsf{f} \in \mathcal{F}$ and $\tau_0 \subseteq^{\mathsf{fin}} \tau$ then there is $\mathsf{g} \in \mathcal{F}$ with $\mathsf{f} \subseteq \mathsf{g}$ and $\tau_0 \subseteq \mathrm{dom}(\mathsf{g})$

M is a **pseudomodel** for T if there is a forth system $\mathcal{F}$ from τ to τ^* such that for every $\mathsf{f} \in \mathcal{F}$, $\mathsf{M} \models \mathsf{f}''_* \mathsf{T}^{\mathrm{dom}(\mathsf{f})}$.

Pseudo-models: a picture

The notion of pseudomodel deals with

- localizing in coherent ways (sheaflike construction) the notion of being a model,



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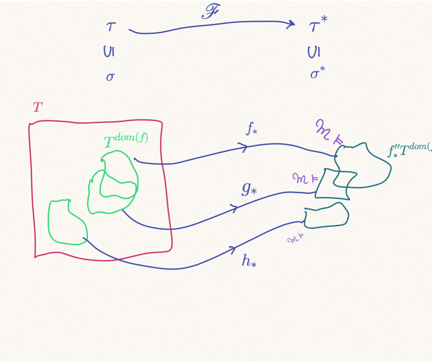

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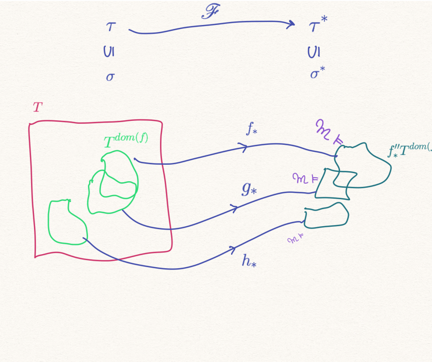

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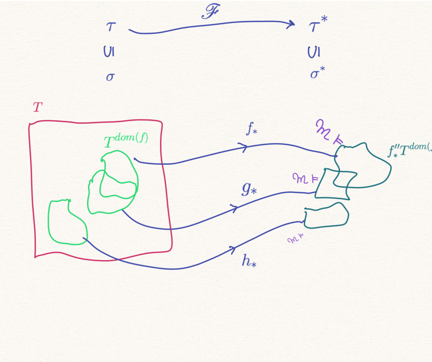

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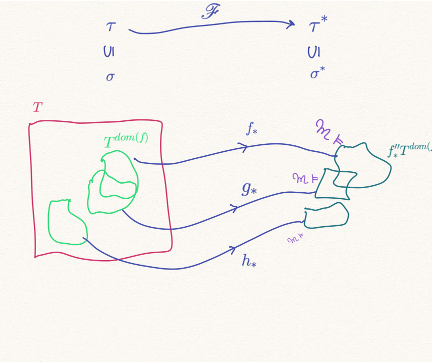

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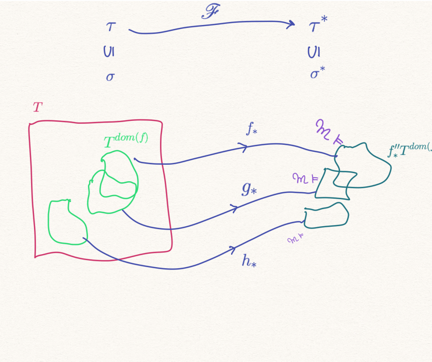
- **Motto:** forth-systems between vocabularies $\equiv$ forcing notions for virtuality



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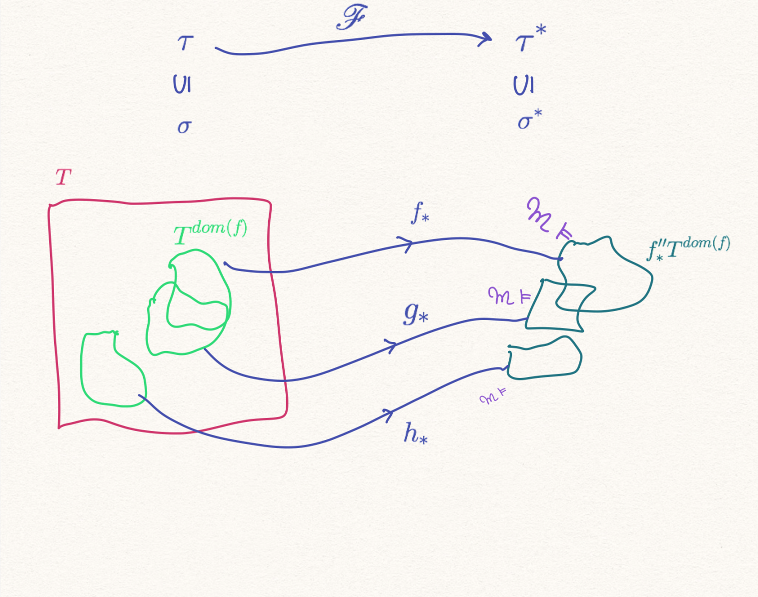
Pseudomodels



Virtualization of a Logic

A related notion: the virtualization of a logic. Using forth-systems **for models** (and not for vocabularies, as above).
An $\mathcal{L}$-**forth system** $\mathcal{P}$ from M to N (both τ – structures) is a collection of $\mathcal{L}$-elementary embeddings with the "forth property":

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This is equivalent to playing the classical Ehrenfeucht-Fraïssé game but with ANTI picking only challenges "from the left" (from M). [BDGM] use this to get Löwenheim-Skolem-Tarski style characterizations of virtual cardinals: **the existence of a virtual elementary embedding** $f : M \to N$ is equivalent to the existence of a forth system from M to N or that N satisfies the **virtualized logic** theory of M (or ISO has a winning strategy in the half (virtual) game)…

A direction worth looking at: L^1_θ for θ strongly compact

Shelah has been able to extract interesting model theory from the blend of the definition of L^1_θ **under the additional assumption that** θ is a strongly compact cardinal:

- A "Keisler-Shelah"-like theorem (L^1_θ-elementarily equivalent models have isomorphic iterated ultrapowers)
- Special models (unions of ω-chains of iterated ultrapowers are unique…giving easier proofs of Craig (essentially, showing Robinson and using compactness).
- Connections to stability theory.

The methods are connected with Malliaris-Shelah's constructions and also with careful use of saturation, not unlike the use of forth models in [BDGM].

Virtualizing L^1_κ, $\mathsf{L}^{1,c}_\kappa$, …

There are at least two competing virtualizations of these logics:

- Use the definition from [BDGM]…but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings…

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- Use the definition from [BDGM]... but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings...
- Use a "virtualized" version of the Shelah (or the Cartagena) game $\eth^\beta_\theta$, $\eth^{\beta,c}_\theta$...

Both virtualized versions are essentially existential closures of the logics. They would give rise to two competing notions of virtual embeddings (or different notions of genericity!). So... which one?

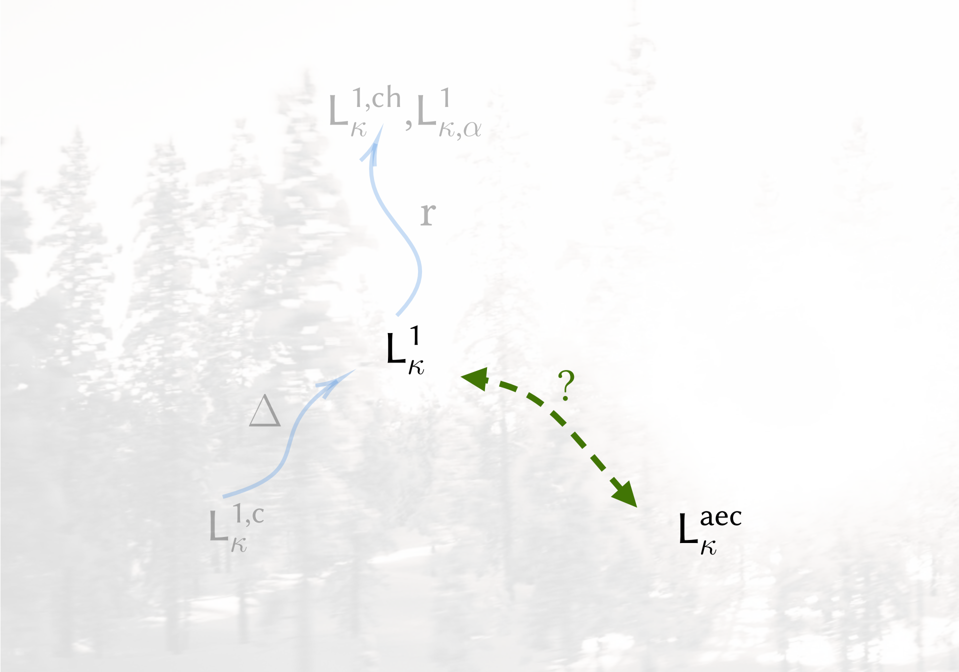
Delayable, virtually delayable...

Definition

A cardinal κ is a delayable cardinal if it is a compactness cardinal for the second-order version of Shelah's logic L^2_κ. It is a virtually delayable cardinal if it is a pseudo-compactness cardinal for L^2_κ.
If we replace L^2_κ by $\mathsf{L}^{2,c}_\kappa$ we get the corresponding two notions of Cart-delayable cardinal and virtually Cart-delayable cardinal.

1. Where are these cardinals located? What kind of reflection properties do they capture?
2. The deeper issue is: what kind of virtuality do they actually correspond to? What version of forth-systems?

Plan

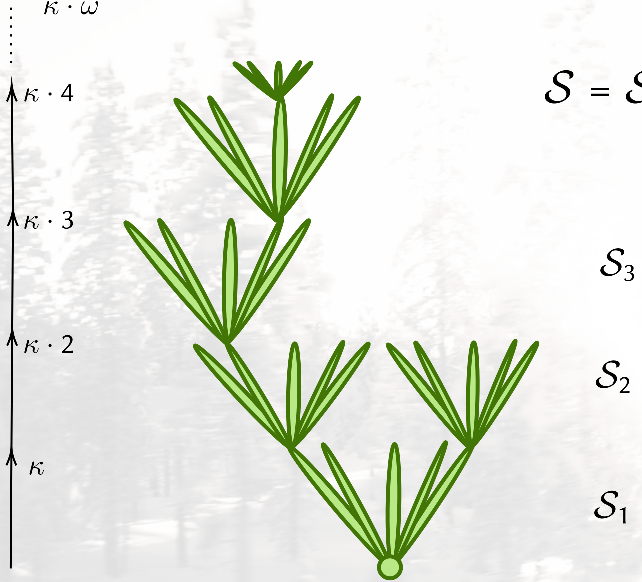

$\mathbb{L}^{1,ch}_\kappa$, $\mathbb{L}^1_{\kappa,\alpha}$
r
$\mathbb{L}^1_\kappa$
?
Δ
$\mathbb{L}^{1,c}_\kappa$
$\mathbb{L}^{aec}_\kappa$

The canonical tree of an a.e.c.

This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_{\mathsf{n}} := \{\mathsf{M} \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_{\mathsf{M}}$ of length n, M has universe $\left\{\mathsf{a}^*_\alpha \mid \alpha \in \mathsf{S}_{\bar{\alpha}[\mathsf{M}]}\right\}$ and $\mathsf{m} < \mathsf{n} \Rightarrow \mathsf{M} \restriction \mathsf{S}_{\bar{\alpha}\restriction \mathsf{m}[\mathsf{M}]} \prec_{\mathcal{K}} \mathsf{M}\Big\}$ (and $\mathcal{S}_0 = \left\{\mathsf{M}_{\mathsf{empt}}\right\}$),
- $\mathcal{S} = \mathcal{S}_{\mathcal{K}} := \bigcup_{\mathsf{n}} \mathcal{S}_{\mathsf{n}}$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalenty under $\subseteq$).

$\mathcal{S}(\mathcal{K})$



$\mathcal{S} = \mathcal{S}(\mathcal{K})$

Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

  $$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

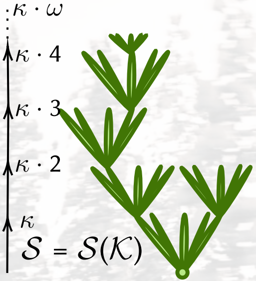
  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

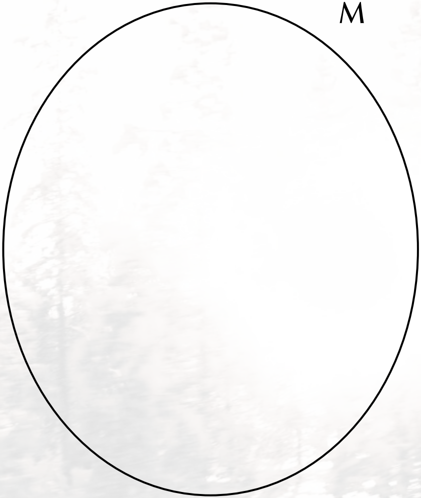
  $$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

  $$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

Testing the class against the tree - Does $M \in \mathcal{K}$?





So we have <u>sentences</u> $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.

Let us take a closer look at low levels:

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When does $M \models \varphi_{1,0}$?

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That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

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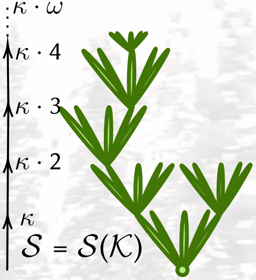
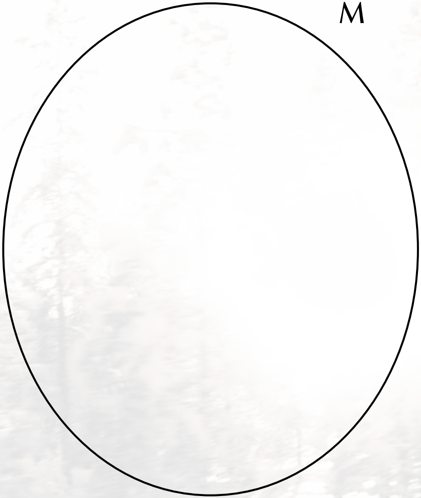
This is slightly more complicated to unravel:

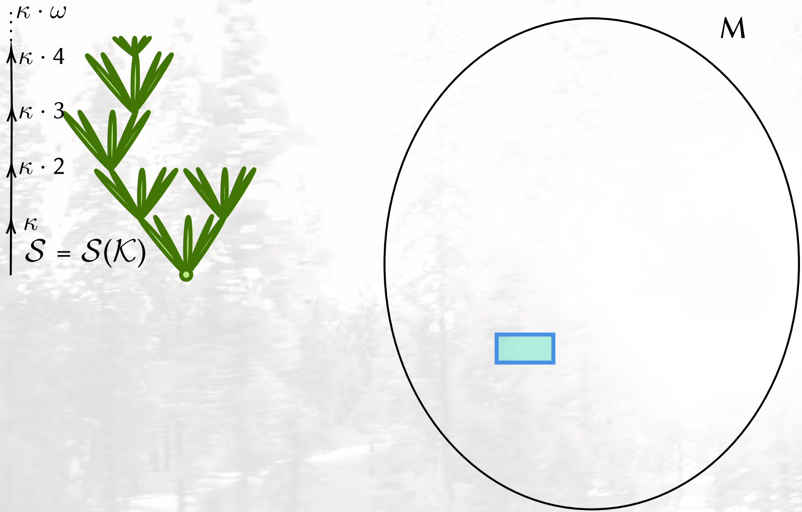
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$
For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z…
for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

The mezcal test - Does $M \in \mathcal{K}$?

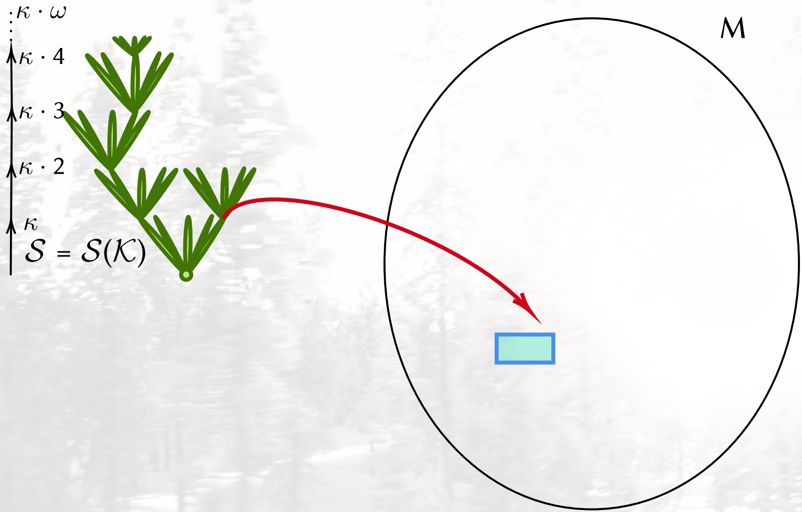




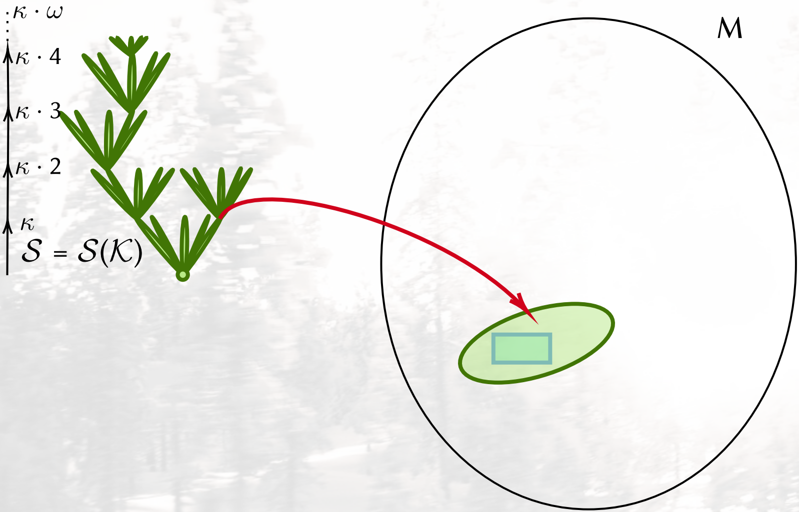
The mezcal test - Does $M \in \mathcal{K}$?



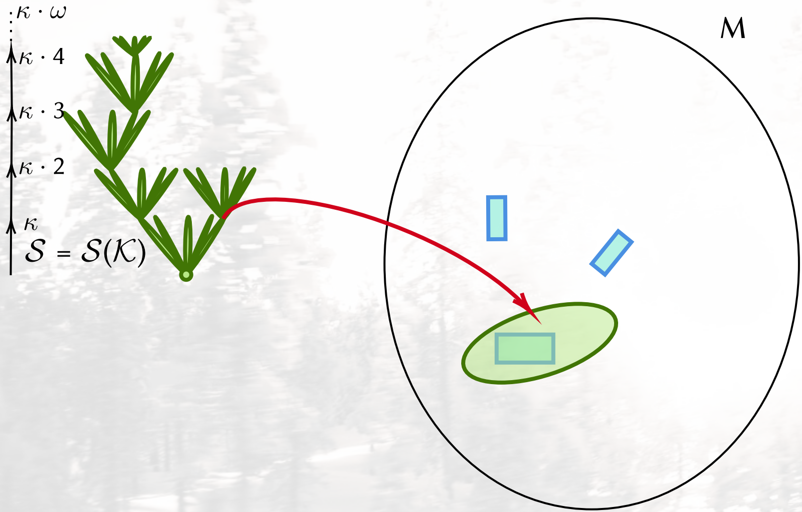
The mezcal test - Does $M \in \mathcal{K}$?



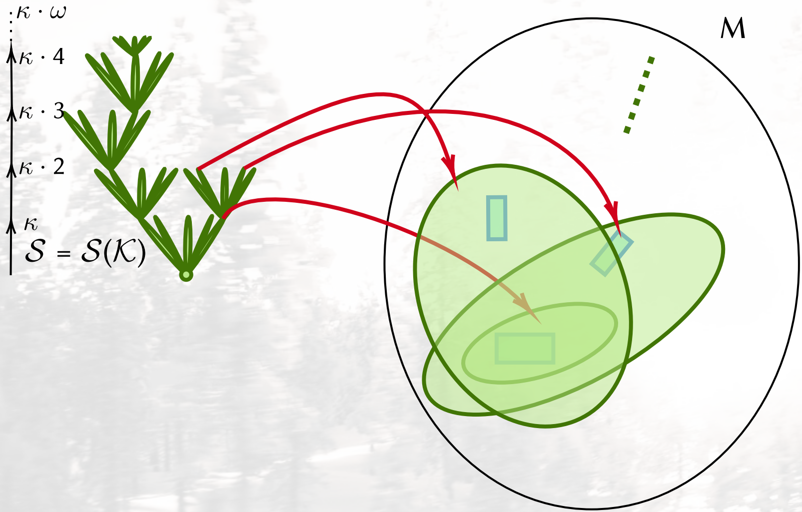
The mezcal test - Does $M \in \mathcal{K}$?



The mezcal test - Does $M \in \mathcal{K}$?



The mezcal test - Does $M \in \mathcal{K}$?



Theorem

$M \in \mathcal{K}$ implies $M \models \varphi_{\gamma,0}$ for each $\gamma < \lambda^+$

Theorem
$M \models \varphi_{\beth_2(\kappa)^+ + 2, 0}$ *implies* $M \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; the partition property due to Komjath and Shelah is the key…

The same partition property that worked for Väänänen and Velickovic's reduction of the game!

The tree property enables us to "reconstruct" M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$).
With this we can

- define "quantificational depth" of an aec (variants of Baldwin-Shelah (building on Mekler and Eklöf) give examples of high quantificational depth)…
- get definability of the "strong submodel relation" $\prec_{\mathcal{K}}$ …and genuine variants of a Tarski-Vaught test
- a grip on biinterpretability of AECs…

The End (Matta: The Integral of Silence)

¡Gracias! Diosï meyamu! Fié nzhinga!