Una mirada al forcing en haces, con aplicaciones en cuántica

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

Seminario Mundo/Lógicas/Modelos
UNAL - 29 de agosto de 2023

Contents

As an introduction / Disclaimer

I adapt part of a lecture given in Brazil in December 2018 (Cantor Meets Robinson) centered on model-theoretic forcing.
The audience here is more focused on different issues: sheaf-forcing.
You may consult in my webpage the other part of the lecture, if interested (Nájar might look at connections between forcing and abstract elementary classes).

Chapter 1

Forcing on Sheaves/Topoi

A unified perspective

We have seen so far:

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

A unified perspective

We have seen so far:

- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

A unified perspective

We have seen so far:

- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.
- The notion of genericity in Fraïssé/Hrushovski constructions and its connection with Model Companions (and thereby with Model Theoretic Forcing).

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

A unified perspective

We have seen so far:

- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.
- The notion of genericity in Fraïssé/Hrushovski constructions and its connection with Model Companions (and thereby with Model Theoretic Forcing).
- A more contemporary take on model theoretic forcing, in "Reflection Classes", allowing to capture abstract forms of homogeneity through limit (brimmed) models.

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

Extended Objects / Variable Objects

Objects in the world present themselves as extended in time (or in other classical (or non-classical) “categories”):

- Physical objects, individuals, etc.

Leibniz, Peirce, Husserl, etc.

Extended Objects / Variable Objects

Objects in the world present themselves as extended in time (or in other classical (or non-classical) “categories”):

- Physical objects, individuals, etc.
- Particles, even neutrinos (for some particles, order of 10^{-20} seconds, yet still “time”)

Leibniz, Peirce, Husserl, etc.

Extended Objects / Variable Objects

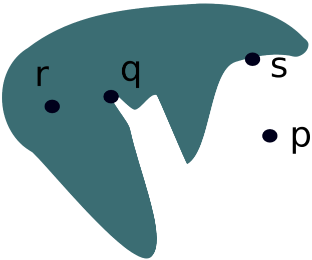
Objects in the world present themselves as extended in time (or in other classical (or non-classical) "categories"):

- Physical objects, individuals, etc.
- Particles, even neutrinos (for some particles, order of 10^{-20} seconds, yet still "time")
- Concepts? Thoughts? Ideas? Visualizations? Perceptions?

Leibniz, Peirce, Husserl, etc.

Yet logic (at the limit) is "too rough"

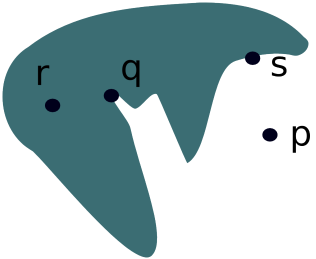
(Really, classical logic.)



- For p and r the predicate “is in the green zone” is clear - classical logic “agrees” with perception.

Yet logic (at the limit) is "too rough"

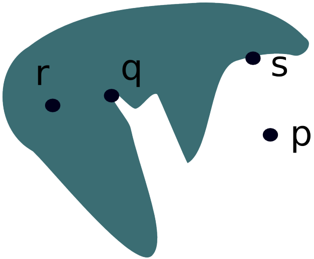
(Really, classical logic.)



- For p and r the predicate “is in the green zone” is clear - classical logic “agrees” with perception.
- For q and s (at “limit situations”) classical logic forces one to make a decision (open, closed green zone, etc.).

Yet logic (at the limit) is "too rough"

(Really, classical logic.)



- Perception does not follow classical logic.

Physics, geometry, and "limit" phenomena

As we know since the late 1920's, Physics (wave models, quantum phenomena of "undecidability" or "uncertainty", noncommutativity of operators corresponding to formalizations of observability, etc.) has the kind of "limit phenomena" that may call for a logic of variable entities.

Physics, geometry, and "limit" phenomena

As we know since the late 1920's, Physics (wave models, quantum phenomena of "undecidability" or "uncertainty", noncommutativity of operators corresponding to formalizations of observability, etc.) has the kind of "limit phenomena" that may call for a logic of variable entities.

Algebraic geometry of the postwar period (Leray, Cartan, Weil, and then Grothendieck reflects this same "shift of perspective": sheaves, sites, topoi.)

Instant velocity / Paradigm change

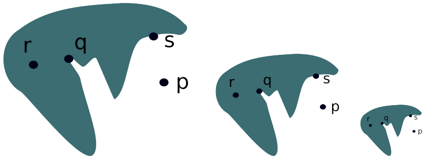
Instant velocity has exactly the same behavior as "the color of point": it really is an abstraction of a property of neighborhoods.
Excluded middle may be dropped!
The strong paradigm becomes Truth Continuity.

Instant velocity / Paradigm change

Instant velocity has exactly the same behavior as "the color of point": it really is an abstraction of a property of neighborhoods.
Excluded middle ~~may~~ HAS to be dropped!
The strong paradigm becomes Truth Continuity.

Truth Continuity

If an individual (an entity, a particle, etc.) has some property on some point of its domain of extension, there has to be a neighborhood of this point in this domain in which this property holds of all points.

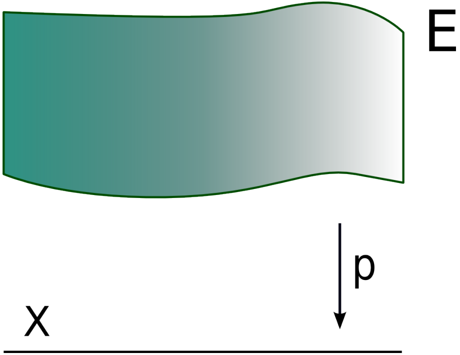


Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.



Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

The previous conditions imply various things:

Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

The previous conditions imply various things:

- The topology induced on the fibers $p^{-1}(a) \subset E$ is discrete, for every $a \in X$,

Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

The previous conditions imply various things:

- The topology induced on the fibers $p^{-1}(a) \subset E$ is discrete, for every $a \in X$,
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),

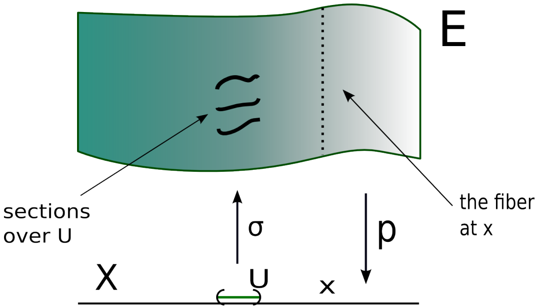
Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

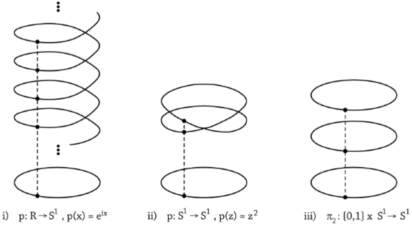
The previous conditions imply various things:

- The topology induced on the fibers $p^{-1}(a) \subset E$ is discrete, for every $a \in X$,
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),
- If two sections σ, τ coincide at a point a then there exists an open set $U \ni a$ such that $\sigma \restriction U = \tau \restriction U$

Sections - objects



Classical and not[1]



Cildo Meireles - Fontes

[1] Caicedo: Lógica de los Haces de Estructuras - Revista Academia Colombiana de Ciencias, 1995

A little history

Sheaves over topological spaces go back to **H. Weyl** (1913), in his work on Riemann surfaces.
They "reappear" strongly in **Cartan**'s seminar (1948-1952) and then catch flight with the French Algebraic Geometry School of the Postwar (**Serre**, **Leray**, etc.).
Weil: Séminaire de géométrie algébrique: study of the zeta function on finite fields.
Finally, **Grothendieck** generalizes further the frame (to sites = small categories endowed with "Grothendieck topologies").
Deligne then proves Weil's conjectures.

Sheaves of Structures

A sheaf of structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(a)$ ($a \in X$), a structure

$$\mathfrak{A}_a = (E_a, (R_i^a)_i, (f_j^a)_j, (c_k^a)_k,)$$

such that $E_a = p^{-1}(a)$, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^{\mathfrak{A}_x}$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^{\mathfrak{A}_x}$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^{\mathfrak{A}_x}$ is a continuous global section

Truth Continuity?

Fact

For all atomic formulas $\varphi(v)$ *we have that*

$$\mathfrak{A}_x \models \varphi(\sigma(x)) \textit{ iff } \exists U \ni x \forall y \in U \Big(\mathfrak{A}_y \models \varphi(\sigma(y)) \Big)$$

This also holds for positive Boolean combinations of atomic formulas.

However, this fails for negations!

Truth Continuity?

Fact

For all atomic formulas $\varphi(v)$ *we have that*

$$\mathfrak{A}_x \models \varphi(\sigma(x)) \textit{ iff } \exists U \ni x \forall y \in U \Big(\mathfrak{A}_y \models \varphi(\sigma(y)) \Big)$$

This also holds for positive Boolean combinations of atomic formulas.
However, this fails for negations!
The solution to this failure is to switch to an emphasis on forcing.

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

$$\mathfrak{A}_x \models \varphi(\sigma(x))$$

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

$$\mathfrak{A}_x \models \varphi(\sigma(x))$$

$$\mathfrak{A} \Vdash_x \varphi(\sigma)$$

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

$$\mathfrak{A}_x \models \varphi(\sigma(x))$$

$$\mathfrak{A} \Vdash_x \varphi(\sigma)$$

$$\mathfrak{A} \Vdash_U \varphi(\sigma)$$

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

$$\mathfrak{A}_x \models \varphi(\sigma(x))$$

$$\mathfrak{A} \Vdash_x \varphi(\sigma)$$

$$\mathfrak{A} \Vdash_U \varphi(\sigma)$$

How do we compare them? Before diving into the definitions of the forcing notions, notice that the first one is pointwise while the second one is local. Also notice that satisfaction in $\mathfrak{A}_x$ is about values of sections at x (the $\sigma(x)$) whereas pointwise (over x) or local forcing (over U) are about the whole section σ defined on U.

Truth continuity - I

Given a formula $\varphi(v)$ of L_t, we define its forcing by $\mathfrak{A}$ at $a \in X$ in such a way that

Truth continuity - I

Given a formula $\varphi(v)$ of L_t, we define its forcing by $\mathfrak{A}$ at $a \in X$ in such a way that

if $\mathfrak{A} \Vdash_a \varphi[\sigma(a)]$ then there exists an open neighborhood U of x such that for every $b \in U$ we also have $\mathfrak{A} \Vdash_b \varphi[\sigma(b)]$.

Truth continuity - I

Given a formula $\varphi(v)$ of L_t, we define its forcing by $\mathfrak{A}$ at $a \in X$ in such a way that

if $\mathfrak{A} \Vdash_a \varphi[\sigma(a)]$ then there exists an open neighborhood U of x such that for every $b \in U$ we also have $\mathfrak{A} \Vdash_b \varphi[\sigma(b)]$.

Sections are the new objects: formulas $\varphi(v_1, v_2, \cdots)$ will be "evaluated" by "replacing" v_i by a section σ_i or by its value at an element x of X, $\sigma_i(x)$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.
- $\mathfrak{A} \Vdash_x (\varphi \wedge \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ and $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x (\varphi \vee \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.
- $\mathfrak{A} \Vdash_x (\varphi \wedge \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ and $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x (\varphi \vee \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x \neg\varphi \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \nVdash_y \varphi$.
- $\mathfrak{A} \Vdash_x (\varphi \to \psi) \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \Vdash_y \varphi$ implies that $\mathfrak{A} \Vdash_y \psi$.

Forcing $\neg, \to, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.
- $\mathfrak{A} \Vdash_x (\varphi \wedge \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ and $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x (\varphi \vee \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x \neg\varphi \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \nVdash_y \varphi$.
- $\mathfrak{A} \Vdash_x (\varphi \to \psi) \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \Vdash_y \varphi$ implies that $\mathfrak{A} \Vdash_y \psi$.
- $\mathfrak{A} \Vdash_x \exists v \varphi(v, \vec{\sigma}) \Leftrightarrow$ there exists some σ defined at x such that $\mathfrak{A} \Vdash_x \varphi[\sigma, \vec{\sigma}]$.

Forcing $\neg, \to, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.
- $\mathfrak{A} \Vdash_x (\varphi \wedge \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ and $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x (\varphi \vee \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$.
- $\mathfrak{A} \Vdash_x \neg\varphi \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \nVdash_y \varphi$.
- $\mathfrak{A} \Vdash_x (\varphi \rightarrow \psi) \Leftrightarrow$ for some open $U \ni x$, for <u>every</u> $y \in U$, $\mathfrak{A} \Vdash_y \varphi$ implies that $\mathfrak{A} \Vdash_y \psi$.
- $\mathfrak{A} \Vdash_x \exists v \varphi(v, \vec{\sigma}) \Leftrightarrow$ there exists some σ defined at x such that $\mathfrak{A} \Vdash_x \varphi[\sigma, \vec{\sigma}]$.
- $\mathfrak{A} \Vdash_x \forall v \varphi(v, \vec{\sigma}) \Leftrightarrow$ for some $U \ni x$, for <u>every</u> $y \in U$ and <u>every</u> σ defined on y, $\mathfrak{A} \Vdash_y \varphi[\sigma, \vec{\sigma}]$.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Truth continuity - II

A semantics can also be defined *directly* over open sets:

$$\mathfrak{A} \Vdash_U \varphi[\sigma],$$

where U is an open set in the domain of σ.

Definition

$\mathfrak{A} \Vdash_U \varphi[\sigma]$ if and only if for every $x \in U$, $\mathfrak{A} \Vdash_x \varphi[\sigma(x)]$.

Chapter 2

Genericity

Generic Filters

Definition

Given $\mathfrak{A}$ a sheaf of structures over X, a generic filter $\mathbb{F}$ for $\mathfrak{A}$ is a filter of open sets of X such that

- for every $\varphi(\sigma)$ and every σ defined on $U \in \mathbb{F}$, there is some $W \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_W \varphi(\sigma)$ or $\mathfrak{A} \Vdash_W \neg\varphi(\sigma)$
- for every σ defined on $U \in \mathbb{F}$, for every $\varphi(u, \sigma)$, if $\mathfrak{A} \Vdash_U \exists u \varphi(u, \sigma)$, then there exists $W \in \mathbb{F}$ and μ defined on W such that $\mathfrak{A} \Vdash_W \varphi(\mu, \sigma)$

For some topological spaces, this definition of genericity of a filter may be made more purely topological/geometrical (and less dependent on formulas and forcing). However, in the general case, this is not necessarily possible - and we must rely on this logical definition.

Existence - generic models

Fact

Generic filters exist.

Definition (Generic Models)

Given a generic filter $\mathbb{F}$ and $\mathfrak{A}(U) = \{\sigma | \mathrm{dom}(\sigma) = U\}$, let

$$\mathfrak{A}[\mathbb{F}] = \lim_{U \in \mathbb{F}} \mathfrak{A}(U) = \bigsqcup_{U \in \mathbb{F}} \mathfrak{A}(U) / \sim_{\mathbb{F}}$$

where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \upharpoonright W = \mu \upharpoonright W$.
Also,

Existence - generic models

Fact

Generic filters exist.

Definition (Generic Models)

Given a generic filter $\mathbb{F}$ and $\mathfrak{A}(U) = \{\sigma | \mathrm{dom}(\sigma) = U\}$, let

$$\mathfrak{A}[\mathbb{F}] = \lim_{U \in \mathbb{F}} \mathfrak{A}(U) = \bigsqcup_{U \in \mathbb{F}} \mathfrak{A}(U) / \sim_{\mathbb{F}}$$

where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \restriction W = \mu \restriction W$. Also,

- $(\sigma_1/\sim_{\mathbb{F}}, \dots, \sigma_n/\sim_{\mathbb{F}}) \in R^{\mathfrak{A}[\mathbb{F}]} \Leftrightarrow \exists U \in \mathbb{F} (\sigma_1, \dots, \sigma_n) \in R^{\mathfrak{A}(U)}$
- $f^{\mathfrak{A}[\mathbb{F}]}(\sigma_1/\sim_{\mathbb{F}}, \dots, \sigma_n/\sim_{\mathbb{F}}) = f^{\mathfrak{A}(U)}(\sigma_1, \dots, \sigma_n)/\sim_{\mathbb{F}}$

Limits

Theorem (A classical Generic Model Theorem)

Let $\mathbb{F}$ *be a generic filter for a sheaf of topological structures* $\mathfrak{A}$ *over* X. *Then*

$$\begin{aligned} \mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) &\iff \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma). \end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models. The model $\mathfrak{A}[\mathbb{F}]$ can be regarded as a "fiber at ∞". This may be made precise by extending X by one new point ∞, adding the generic model as the new fiber over ∞ and extending the sections by

$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models. The model $\mathfrak{A}[\mathbb{F}]$ can be regarded as a "fiber at ∞". This may be made precise by extending X by one new point ∞, adding the generic model as the new fiber over ∞ and extending the sections by

$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

The fiber "at infinity"

The model $\mathfrak{A}[\mathbb{F}]$ can be regarded as a "fiber at ∞". This may be made precise by extending X by one new point ∞, adding the generic model as the new fiber over ∞ and extending the sections by

$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

Other applications of the GMTh

- Kripke models - generalized semantics
- Set-theoretic forcing
- Robinson's Joint Consistency Theorem (=Amalgamation over Models)
- Various Omitting Types Theorems (Caicedo, Brunner-Miraglia)
- Control over new kinds of limit models

Chapter 3

Continuous sheaf forcing

Sheaves of Hilbert Spaces

Why?


(Geraldo Barros)

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

THE PROBLEM OF A MODEL THEORY FOR HILBERT SPACES

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.
We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).
Moreover, we want the logic to "keep track" of (say) the distance to a projection $p(\nu)$, the convergence of a sequence in H, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.
That is the rôle of Continuous Model Theory.

Sheaves of Metric Structures

A sheaf of metric structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0, 1])$$

such that $E_x = p^{-1}(x)$, (E_x, d_x) is a complete bounded metric space of diameter 1, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^x$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^x$ is a continuous global section

(further requirements on moduli of uniform continuity)

Sheaves of Metric Structures

A sheaf of metric structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0,1])$$

such that $E_x = p^{-1}(x)$, (E_x, d_x) is a complete bounded metric space of diameter 1, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^x$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^x$ is a continuous global section
- The premetric $d^{\mathfrak{A}} := \bigcup_{x \in X} d_x : \bigcup_{x \in X} E_x^2 \to [0,1]$ is a continuous function.

(further requirements on moduli of uniform continuity)

Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
Here, in "CFO" logic, the semantics is defined on conditions of the form

$$\varphi(x) < \varepsilon, \varphi(x) \leq \varepsilon, \cdots$$

Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
Here, in "CFO" logic, the semantics is defined on conditions of the form

$$\varphi(x) < \varepsilon, \varphi(x) \leq \varepsilon, \cdots$$

Negation in continuous, metric logic, is weak: the semantics really treats $\leq$ and $\geq$ as "negations" of each other...

Pointwise forcing

With M. Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) > \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$

Pointwise forcing

With M. Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) > \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.

Pointwise forcing

With M. Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) > \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x 1 \dot{-} \varphi < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi > 1 \dot{-} \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi < \varepsilon$ iff and only if one of the following holds:
  - $\mathfrak{A} \Vdash_x \varphi < \psi$
  - $\mathfrak{A} \nVdash_x \varphi < \psi$ and $\mathfrak{A} \nVdash_x \varphi > \psi$
  - $\mathfrak{A} \Vdash_x \varphi > \psi$ and $\mathfrak{A} \Vdash_x \varphi < \psi + \epsilon$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi > \varepsilon$ iff $\mathfrak{A} \Vdash_x \varphi > \psi + \varepsilon$
- $\dots$

Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.

Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.
- $\mathfrak{A} \Vdash_x \inf_s \varphi(s) > \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) > \varepsilon + \delta_x$

Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.
- $\mathfrak{A} \Vdash_x \inf_s \varphi(s) > \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) > \varepsilon + \delta_x$
- $\mathfrak{A} \Vdash_x \sup_s \varphi(s) < \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) < \varepsilon - \delta_x$
- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) > \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) > \varepsilon$.

A METRIC ON SECTIONS? (NOT YET)

So far so good, but we have (for the time being) lost the metric on the sections (so, the corresponding presheaves $\mathfrak{A}(U)$ are still missing the "metric" feature - they do not live in the correct category yet).

- Sections have different domains
- Triangle inequality is tricky
- Restrict to sections with domains in a filter of open sets
- But the ultralimit (even in that case) could fail to be complete!

Rather... a pseudometric

Fix F a filter of open sets of X. For all sections σ and μ with domain in F define

$$F_{\sigma\mu} = \{U \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | U \in F\}.$$

Then the function

$$\rho_F(\sigma, \mu) = \inf_{U \in F_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric on the set of sections with domain in F.
In some cases we may get completeness of the induced metric:

Rather... a pseudometric

Fix F a filter of open sets of X. For all sections σ and μ with domain in F define

$$F_{\sigma\mu} = \{U \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | U \in F\}.$$

Then the function

$$\rho_F(\sigma, \mu) = \inf_{U \in F_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric on the set of sections with domain in F.
In some cases we may get completeness of the induced metric:

Lemma (Ochoa, V.)

Let $\mathfrak{A}$ be a sheaf of metric structures defined over a regular topological space X. Let F be an ultrafilter of regular open sets. Then, the metric induced by ρ_F on $\mathfrak{A}[F]$ is complete.

Other solutions include just working with pseudometrics and give up completeness, or even working with more general frameworks.

Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathfrak{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists\delta < \varepsilon \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists\varepsilon > \delta \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma))$

Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathfrak{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists \delta < \varepsilon \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists \varepsilon > \delta \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma))$

There is an involved, equivalent, inductive definition. We also have $\mathfrak{A} \Vdash_U \inf_\sigma(1 \dot{-} \varphi(\sigma)) > 1 \dot{-} \varepsilon \iff \mathfrak{A} \Vdash_U \sup_U \varphi(\sigma) < \varepsilon$, and a maximal principal principle (existence of witnesses of sections).

Metric Generic Model / Forcing Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathfrak{A}[\mathsf{F}]$.

Metric Generic Model / Forcing Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathfrak{A}[F]$.

Theorem (Metric GMTh)

Let F *be a generic filter on* X, $\mathfrak{A}$ *a sheaf of metric structures on* X *and* $\sigma_1, \cdots, \sigma_n$ *sections. Then*

1. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) < \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) < \varepsilon$
2. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) > \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) > \varepsilon$

A Metric Sheaf for noncommuting observables with continuous spectra

Really, a metric sheaf space for a free particle:

Definition

The triple $\mathfrak{A}_{\mathrm{cont}} = (E, X, \pi)$ where

- $X = \mathbb{R}^{+}$ is the base space with the product topology.
- For $\tau \in X$ we let E_τ be a two sorted metric model where
  - $\mathcal{U}_\tau$ and $\mathcal{V}_\tau$ span the universe for each sort.
  - Every sort has is a metric space with the metric induced by the norm in $\mathcal{L}^2(\mathbb{R})$.
  - Every sort is a model in the language of a vector space, with symbols for the binary transformation $\langle , \rangle_{\mathcal{V}}$ and $\langle , \rangle_{\mathcal{U}}$, to be interpreted such that

A Metric Sheaf for noncommuting observables with continuous spectra

- $$\langle q(x_0 - x)\phi_{(\tau,t_1)}(x_0 - x), r(x_1 - x)\phi_{(\tau,t_1)}(x_1 - x)\rangle_{\mathcal{U}} = q(x_0 - x_1)r(x_0 - x_1)\phi_{(\tau,t_1+t_2)}(x_0 - x_1) \quad (1)$$

  $$\langle q(p_0 - p)\phi_{1/(\tau,t_1)}(p_0 - p), r(p_1 - p)\phi_{1/(\tau,t_1)}(p_1 - p)\rangle_{\mathcal{V}} = q(p_0 - p_1)r(p_0 - p_1)\phi_{1/(\tau,t_1+t_2)}(p_0 - p_1) \quad (2)$$

- function symbols for FT and FT^{-1} (Fourier transforms between the operators)
- The sheaf is constructed as the disjoint union of fibers: $E = \sqcup_{\tau \in X} E_\tau$
- Sections are defined such that if $\tau \in U \subset X$,

  $$\sigma_{q,x_0,p_0,t}(\tau) = \left(q(x - x_0\phi_{(\tau,t)}(x, x_0)\,,\; q(p - p_0)\phi_{1/(\tau,t)}(p, p_0)\right).$$
- π, the local homeomorphism, is given by $\pi(\psi) = \tau$ if $\psi \in E_\tau$.

Remarks

- The binary transformations $\langle , \rangle_{\mathcal{U}}$ and $\langle , \rangle_{\mathcal{V}}$ are not the objects usually defined as the inner product in a Hilbert space. Instead, they are our representation for the physical inner product as defined by Dirac in each sort.
- We are interested in two kinds of generic metric models:
  1. In the first kind we look at generic models that capture the limit of vanishing τ, for which we take the nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$. From the structure of the sheaf defined above, limit elements in the generic model coming from the $\mathcal{U}$ sort with $t = 0$ must approach Dirac's delta in position.
  2. On the other hand, the generic metric model we obtain by taking the nonprincipal ultrafilter induced by the family of open regular sets $\{(n, \infty) : n \in \mathbb{N}\}$ must contain limit elements that represent Dirac's distributions in momentum space.

Whence all this?

- Laurent Schwartz's work on distributions
- Schwartz spaces for position and momentum operators

Back to the Schwartz Space - and to the Sheaf construction

One motivation: Dirac's distribution in $\mathcal{L}^2(\mathbb{R})$:

$$\lim_{\tau\to 0}\frac{1}{\tau\sqrt{\pi}}e^{-x^2/\tau^2}=\delta(x) \tag{3}$$

(with the limit taken in the sense of distributions). This suggests that an *imperfect*[2] representation $\phi_\tau(x, x_0)$ for the physical vector state $|x_0\rangle$ in $\mathcal{L}^2(\mathbb{R})$ is

$$\phi_\tau(x, x_0)=\frac{1}{\tau\sqrt{2\pi\hbar}}e^{-(x-x_0)^2/2\hbar\tau^2}. \tag{4}$$

The family of elements $\{\phi_\tau(x, x_0)\}$ is a subset of the Schwartz space and, with the inner product in $\mathcal{L}^2(\mathbb{R})$, we find that

$$\langle\phi_\tau(x, x_0), \phi_\tau(x, x_1)\rangle=\int_{-\infty}^{\infty}dx\phi_\tau(x, x_0)\phi_\tau(x, x_1)=\phi_{\sqrt{2}\tau}(x_1, x_0). \tag{5}$$

[2] In the sense of "up to τ"

Imperfect propagator at the fiber E_τ

After many calculations we get an ugly expression for the imperfect propagator at the fiber E_τ:

$$\begin{aligned}\langle x_1, U(t)x_0\rangle &= \langle \phi_\tau(x, x_1), \phi_{(\tau, it/m)}(x, x_0)\rangle_{\mathcal{U}} && (6)\\ &= \phi_{(\tau, it/m)}(x_1, x_0) && (7)\\ &= \frac{1}{\sqrt{2\pi(\tau^2 + it/m)}} e^{-(x_1 - x_0)^2/2\hbar(\tau^2 + it/m)} && (8)\end{aligned}$$

Letting $\tau \to 0$, we recover the exact form for the quantum mechanical amplitude; (with any nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$).

Imperfect propagator at the fiber E_τ

After many calculations we get an ugly expression for the imperfect propagator at the fiber E_τ:

$$\langle x_1, U(t)x_0\rangle = \langle \phi_\tau(x, x_1), \phi_{(\tau, it/m)}(x, x_0)\rangle_{\mathcal{U}} \quad (6)$$

$$= \phi_{(\tau, it/m)}(x_1, x_0) \quad (7)$$

$$= \frac{1}{\sqrt{2\pi(\tau^2 + it/m)}} e^{-(x_1 - x_0)^2/2\hbar(\tau^2 + it/m)} \quad (8)$$

Letting $\tau \to 0$, we recover the exact form for the quantum mechanical amplitude; (with any nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$). **Thus in the Generic model $\mathfrak{A}[\mathbb{F}]$ we recover the exact propagator as a limit element.**

Conclusions

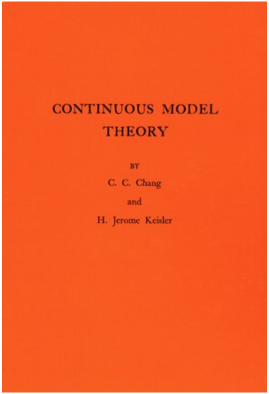
- The classical connection between Robinson forcing and model companions... and Fraïssé / Hrushovski limits
- The long quest by Zilber for Structural Approximation
- Vaught / Harnik: Approximation and Preservation — and possibilities for current work in L^1_κ
- Shelah-Vasey shed light on AECs but also possibly on forcing axioms
- Sheaf forcing seems to unify in a different way (and responds to Zilber's questions)

Continúan los temas… ¡Mil gracias!

Chapter 4

Continuous Model Theory

Continuous Model Theory - Origins



Although the origins of CMTh go back to Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometry recent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Continuous predicates and functions

Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \to M.$$

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in J}, (a_k)_{k \in K}\right)$$

Each function, relation must be endowed with a modulus of uniform continuity.

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \Big(M, d, (f_i)_{i\in I}, (R_j)_{j\in J}, (a_k)_{k\in K}\Big)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0, 1]$, the a_k are distinguished elements of M.
Remember: M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.
- Banach algebras (bounding them).
- Hilbert spaces with inner product as a binary predicate.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.
- Banach algebras (bounding them).
- Hilbert spaces with inner product as a binary predicate.
- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.
- Banach algebras (bounding them).
- Hilbert spaces with inner product as a binary predicate.
- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of C^*-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the t_i are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a \in M} |\varphi^M(a) - \psi^M(a)|$.

The satisfaction relation is defined on conditions rather than on formulas.

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a \in M} |\varphi^M(a) - \psi^M(a)|$.

The satisfaction relation is defined on conditions rather than on formulas.

Conditions are expressions of the form $\varphi(x) \leq \psi(y)$, $\varphi(x) \leq \psi(y)$, $\varphi(x) \geq \psi(y)$, etc.

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a \in M} |\varphi^M(a) - \psi^M(a)|$.

The satisfaction relation is defined on conditions rather than on formulas.

Conditions are expressions of the form $\varphi(x) \leq \psi(y)$, $\varphi(x) \leq \psi(y)$, $\varphi(x) \geq \psi(y)$, etc.

Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every ε, for every connective $f(t_1, \cdots, t_n)$ there exists a connective $g(t_1, \cdots, t_n)$ generated by these four by composition such that $|f(\vec{t}) - g(\vec{t})| < \varepsilon$.

Stability Theory

- Stability (Ben Yaacov, Iovino, etc.),

Stability Theory

- Stability (Ben Yaacov, Iovino, etc.),
- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

Stability Theory

- Stability (Ben Yaacov, Iovino, etc.),
- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory: no analog to Baldwin-Lachlan (no minimality, except some openings by Usvyatsov and Shelah in the context of $\aleph_1$-categorical Banach spaces),
- NO simplicity!!! (Berenstein, Hyttinen, V.),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.
2. Continuous $L_{\omega_1\omega}$. So far, no published results as such. There are however "Lindström theorems" for Continuous First Order due to Caicedo and Iovino.

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.
2. Continuous $L_{\omega_1 \omega}$. So far, no published results as such. There are however “Lindström theorems” for Continuous First Order due to Caicedo and Iovino.
3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. Back to main.